Diffeology, Groupoids & Morita Equivalence

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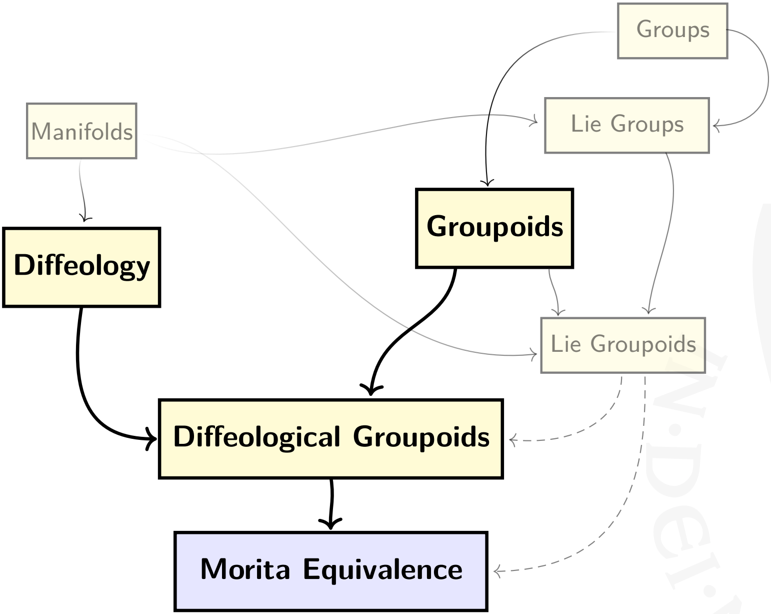

Groups
Lie Groups
Manifolds
Groupoids
Diffeology
Lie Groupoids
Diffeological Groupoids
Morita Equivalence

General Idea: Diffeological Groupoids

- *Diffeological space* = generalised smooth space.
- *Groupoid* = generalised symmetry.
- *Diffeological groupoid* = generalised smooth symmetry.

Used in:

- General relativity
  - Blohmann, Fernandes, Weinstein, arXiv:1003.2857,
  - Jan Głowacki's MSc thesis.
- Singular foliations & algebroids
  - Garmendia, Zambon, arXiv:1803.00896,
  - Androulidakis, Zambon. *Integration of singular subalgebroids*. (In preparation.)
- (Differentiable) Stacks
  - Watts, Wolbert, arXiv:1406.1392.

General Idea: Morita Equivalence

- Origin: Morita equivalence of *rings* (Morita, 1950s):

$$R_1 \simeq_{\mathrm{ME}} R_2 \qquad \text{iff} \qquad \text{equivalent rep. theory.}$$

- Morita equivalence of C*-algebras (Rieffel, 1970s).
- Morita equivalence of *Lie groupoids* (1980s).

General Idea: Morita Equivalence

There are two ways to define Morita equivalence:

- *Internally:* the existence of a special *'bimodule'*.
- *Externally:* as an *isomorphism* in a category.

Morita's Fundamental Theorem (1958)

These definitions are the same.

Our Main Theorem is a version of this:

Our Morita Theorem

A diffeological bibundle between diffeological groupoids is weakly invertible if and only if it is biprincipal.

- Groupoids
- Diffeological Groupoids
- Diffeology
- (Diffeological) Bibundles
- Biprincipality (of Bibundles) (= *'special bimodules'*)
- Weak Invertibility (of Bibundles) (= *'isomorphism in category'*)

Diffeology

- Throughout, fix a set X.

Definition

A *Euclidean domain* is an open subset $U \subseteq \mathbb{R}^m$, for some $m \in \mathbb{N}_{\geqslant 0}$.

Definition

A *parametrisation* is a function $U \longrightarrow X$ defined on a Euclidean domain.

Diffeology

- Determines which parametrisations are *'smooth'*.

Definition

A *diffeology* is a subset $\mathcal{D} \subseteq \mathrm{Param}(X)$, whose elements are called *plots*, satisfying three axioms:

Covering	Constant maps $U \longrightarrow X$ are plots.
Compatibility	Plots can be reparametrised by smooth functions $V \longrightarrow U$.
Locality	If $U \longrightarrow X$ is locally a plot everywhere, then it is a plot.

A pair $(X, \mathcal{D}_X)$ is called a *diffeological space*.

Smooth Maps

- Plots are the 'smooth' functions $U \xrightarrow{\alpha} X$.
- A function $f : X \longrightarrow Y$ gives a composition $U \xrightarrow{\alpha} X \xrightarrow{f} Y$.

Definition

A function $f : (X, \mathcal{D}_X) \longrightarrow (Y, \mathcal{D}_Y)$ is *smooth* if it takes plots to plots:

$$\forall \alpha \in \mathcal{D}_X \text{ we have } f \circ \alpha \in \mathcal{D}_Y.$$

First Examples

- $\mathrm{Param}(X)$ is always a diffeology.
  - Every $Z \longrightarrow X$ is smooth.
- $\mathcal{D}_X = \{$constant parametrisations$\}$ violates Locality.
- $\mathcal{D}_X = \{$*locally* constant$\}$ *does* form a diffeology.
  - Every $X \longrightarrow Y$ is smooth.

Manifolds as Diffeologies

- Every smooth manifold M gets a canonical diffeology $\mathcal{D}_M$.
  - Plots are just the usual smooth functions!
- We get a fully faithful embedding:

$$\mathbf{Manifold} \hookrightarrow \mathbf{Diffeology}.$$

Quotients

- Let $\sim$ be any equivalence relation on X.
- The quotient space $X/\sim$ *always* has a natural diffeology.
- Important example, the *irrational torus*:

$$T_\theta := \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}).$$

  - Is topologically trivial,
  - but carries *non*-trivial diffeological information!

Function Spaces & Convenience

- Let X and Y be diffeological spaces.
- Then $C^\infty(X, Y)$ itself has a natural diffeology!
- More generally, typical ∞-dimensional manifolds all embed fully faithfully into $\mathbf{Diffeology}$.

Convenience Theorem

$\mathbf{Diffeology}$ is complete, cocomplete and Cartesian closed.

Groupoids

The idea: *Groupoid* = group with *partially defined* multiplication.

Definition

A *groupoid* $G \rightrightarrows G_0$ consists of:

- Set of *objects* G_0,
- Set of *arrows* G (between objects).

Arrows $g \in G$ denoted $g : x \to y$, for objects $x, y \in G_0$.
Together with five *structure maps:*

1. Source: $s(g) = x$,
2. Target: $t(g) = y$,
3. Identity arrows: $u(x) = \mathrm{id}_x$,
4. Inverse: $i(g) = g^{-1}$,
5. Multiplication/composition: if $h : y \to z$, then $m(h, g) = h \circ g$.

First Examples

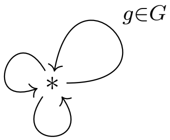
- Unit groupoid: $A \rightrightarrows A$.
  - $G_0 = A$ is an arbitrary set.
  - *Only* identity arrows.
  - All structure maps are id_A, no non-trivial multiplication.
- Pair groupoid: $A \times A \rightrightarrows A$.
  - $G_0 = A$ is an arbitrary set.
  - For every two points $a, b \in A$, there is a *unique* arrow $a \to b$.
  - Denote this arrow by $(b, a) : a \to b$. This means $G = A \times A$.
  - Structure maps:

$$s(b,a) = a, \quad \text{and} \quad t(b,a) = b,$$

$$(c,b) \circ (b,a) := (c,a).$$

First Examples

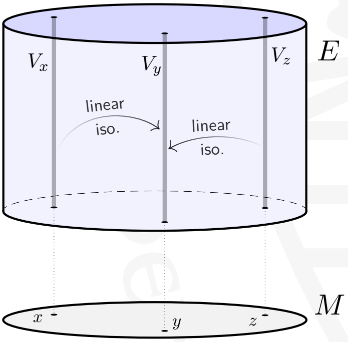
- Any *group* G is also a group*oid* $G \rightrightarrows 1$:
  - $G_0 = 1 = \{*\}$ is the one-point set.
  - Arrows are just the group elements G.
  - Groupoid composition is the group multiplication:



First Examples

- Any vector space V gives the general linear *group* $\mathrm{GL}(V)$.
  - Contains linear isomorphisms $V \longrightarrow V$.
- Now take a collection E of vector spaces V_x, indexed by $x \in M$, e.g. a smooth *vector bundle* $E \longrightarrow M$.

- For each x we still get $\mathrm{GL}(V_x)$.
- But we can also look at linear isomorphisms $V_x \longrightarrow V_y$.
- Thus we get a group*oid*: $\mathrm{GL}(E) \rightrightarrows M$, the *general linear groupoid*.
- If $E \longrightarrow M$ is a smooth vector bundle, it is a *Lie groupoid*.



Diffeological Groupoids

- *Diffeological groupoid* = groupoid + diffeological structure.

Definition

A *diffeological groupoid* is a groupoid $G \rightrightarrows G_0$, where G_0 and G are equipped with diffeologies, such that all structure maps become smooth.

Groupoid Actions

- *Groupoid action* = generalisation of *group* action.
- The idea: a *partially defined* group action.

Definition

A *smooth left groupoid action* of $G \rightrightarrows G_0$ on X consists of:

- A smooth function $l_X : X \to G_0$.
- A function
$$(g, x) \longmapsto g \cdot x \in X,$$
only defined if $s(g) = l_X(x)$.

This has to be smooth in both g and x. We write:

$$G \overset{l_X}{\curvearrowright} X.$$

First Examples

- Groupoid multiplication from the left:

$$(g, h) \longmapsto g \circ h.$$

- Groupoid actions of $G \rightrightarrows 1$ are just group actions.

Definition

We assume a similar definition for *smooth <u>right</u> groupoid actions*:

$$X \ {}^{r_X}\!\curvearrowleft H.$$

Groupoid Bibundles

- The idea: analogue of bimodules for rings/algebras.
- That is: *Bibundle* = 'compatible' smooth left- *and* right actions.

Definition

A (*diffeological*) *bibundle* consists of:

- $G \curvearrowright^{l_X} X$,
- $X \; {}^{r_X}\!\!\curvearrowleft H$,

such that the actions commute. We write:

$$G \curvearrowright^{l_X} X \; {}^{r_X}\!\!\curvearrowleft H.$$

Groupoid Bibundles

- X is a 'bundle' over G_0 and H_0 simultaneously:

$$\begin{tikzcd} G \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & X \arrow[dll, "l_X"] \arrow[drr, "r_X"'] & \curvearrowleft & H \arrow[d, shift left] \arrow[d, shift right] \\ G_0 & & & & H_0, \end{tikzcd}$$

- A bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is a new type of morphism $G \xrightarrow{X} H$.

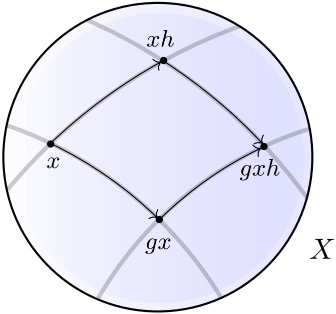
Morita Equivalence

- The idea: existence of a *'nice'* bibundle.

Definition

A bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is called *biprincipal* both actions are *'nice'*; in particular they are *free* and *transitive*.

- Meaning: the actions of G and H *'fill in'* the entire X:



Morita Equivalence

Internal Definition

We say $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are ***Morita equivalent*** if there exists a biprincipal bibundle between them. In that case we write $G \simeq_{\mathrm{ME}} H$.

The Orbit Space

- Important 'quantity' of a groupoid $G \rightrightarrows G_0$: which objects are *'connected'*.

Definition

The *orbit* of an object $x \in G_0$ is:

$$\mathrm{Orb}_G(x) := \{y \in G_0 : \exists x \xrightarrow{g} y\}.$$

The collection of all orbits is denoted G_0/G, the *orbit space*.

- As a quotient, it is naturally a diffeological space.

Morita Equivalence

- The interpretation is that Morita equivalence measures:
  - The *'connectedness'*,
  - but also the *way* in which objects are connected.
- In particular:

Theorem

If $G \simeq_{\mathrm{ME}} H$ then $G_0/G \cong H_0/H$.

- More generally:

  Morita equivalence preserves: *orbit space* + *'isotropy data'*.

First Examples

- Unit groupoids: $A \rightrightarrows A$ and $B \rightrightarrows B$ (recall: only identity arrows):
  - The orbit spaces are just A and B.
  - There is no non-trivial isotropy data,
  - preservation of orbit spaces is all that matters:
$$(A \rightrightarrows A) \simeq_{\mathrm{ME}} (B \rightrightarrows B) \qquad \text{iff} \qquad A \cong B.$$
- Pair groupoids: $A \times A \rightrightarrows A$ (recall: $\exists! a_1 \to a_2$) are *'the most trivial'*:
  - The orbit spaces are trivial,
  - contains only trivial isotropy data, so:
$$(A \times A \rightrightarrows A) \simeq_{\mathrm{ME}} (1 \rightrightarrows 1).$$

First Examples

- Morita equivalence for *groups* $G \rightrightarrows 1$ is just isomorphism.
  - Orbit spaces are trivially preserved: $1 \cong 1$,
  - now isotropy data (i.e. the group structure) is all that matters:

$$(G \rightrightarrows 1) \simeq_{\mathrm{ME}} (H \rightrightarrows 1) \qquad \text{iff} \qquad G \cong H.$$

A More Complicated Example

- Fix a space X that locally looks like U:
  - There exists *local diffeomorphisms* $U \longrightarrow X$ that *cover* X.
  - We call these *'charts'* on X.
  - The collection $\mathscr{A}$ of charts is called an *atlas*.
- Just as for manifolds, we get *transition functions* between charts.
- These collect into a diffeological groupoid:
  - The *transition groupoid*: $\mathbf{Trans}(\mathscr{A})$.
  - This is not generally a Lie groupoid!

Theorem

If $\mathscr{A}_X$ is an atlas for X, and $\mathscr{A}_Y$ is an atlas for Y, then:

$$\mathbf{Trans}(\mathscr{A}_X) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}_Y) \qquad \text{iff} \qquad X \cong Y.$$

The Morita Theorem, Revisited

- We think of bibundles as morphisms $G \xrightarrow{X} H$.
- Composition of $G \xrightarrow{X} H$ and $H \xrightarrow{Y} K$ defined as:

$$X \otimes_H Y := \left(X \times_{H_0}^{r_X, l_Y} Y \right) / H.$$

  - *Hilsum-Skandalis tensor product*,
  - also known as the *balanced tensor product*.
  - Diffeology is *convenient!*

- Gets a canonical bibundle structure:

$$G \curvearrowright X \otimes_H Y \curvearrowleft K,$$

representing an arrow $G \xrightarrow{X \otimes_H Y} K$.

The Morita Theorem, Revisited

- We get a (bi)category of diffeological groupoids and bibundles as morphisms.
- *Weak invertibility* = invertibility of bibundles in this category.
  - Concretely: $G \xrightarrow{X} H$ is *weakly invertible* if $\exists H \xrightarrow{Y} G$ such that

$$\text{“}X \otimes_H Y = \mathrm{id}_G\text{”} \qquad \text{and} \qquad \text{“}Y \otimes_G X = \mathrm{id}_H\text{”}.$$

Morita Theorem

Weak invertibility = *Biprincipality*.

Thanks for your attention!