Master Seminar

Outline of Fourth Lecture on Differentiable Stacks

Nesta van der Schaaf*

20th October 2017

On the 23rd of October, 2017, the fourth and fifth lectures in the series on differentiable stacks will take place. The following is the plan for the fourth lecture.

Outline of the fourth lecture

Recap. We recall some of the material of last week. In the second lecture we embedded the category **Man** into the category **Cat**/**Man**, via the *'manifold stacks'* $\underline{X}$. The point of defining stacks (over **Man**) is to generalise the notion of both manifolds X and quotients X/G to objects in the category **Cat**/**Man**.

In particular, we need the definition of a *quotient stack* $[X/G]$. We recall what its objects and arrows are, and that we can perform *pullbacks* in a quotient stack. (Pullbacks of principal bundles were defined in the third lecture.) We recall that, when the action of G on X is free and proper, then $[X/G] \cong X/G$, showing that $[X/G]$ may serve as a generalisation of the quotient. We show that quotient stacks can be seen as objects in **Cat**/**Man**. For this, first we will recall the definition of a category lying over another category, and then the category of categories over **Man** (as defined in the second lecture).

Groupoid fibrations. The fact that we can perform pullbacks in quotient stacks motivates the definition of *groupoid fibrations*. We define a groupoid as a category where every arrow is an isomorphism. Motivated by quotient stacks, we define a ***groupoid fibration*** $\pi : \mathfrak{X} \to \mathfrak{S}$ as a category $\mathfrak{X}$ lying over a category $\mathfrak{S}$, with the additional properties that the two diagrams

$$\begin{tikzcd}
\exists y \arrow[r, dotted, "\exists"] \arrow[d, dotted, mapsto] & x \arrow[d, mapsto] \\
V \arrow[r] & U
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
y \arrow[rr] \arrow[dd, mapsto] & & x \arrow[dd, mapsto] \\
& z \arrow[ul, dotted, "\exists!"'] \arrow[ur] \arrow[dd, mapsto] & \\
V \arrow[rr] & & U \\
& W \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commute, where $x, y, z \in \mathfrak{X}$ are lying over $U, V, W \in \mathfrak{S}$, respectively, and the arrows in $\mathfrak{X}$ lie over the respective arrows in $\mathfrak{S}$. We prove that the second diagram ensures that the pullback (whose existence is guaranteed by the first diagram) is unique up to unique isomorphism. We define the *fibre* $\mathfrak{X}_U$ of an object U in $\mathfrak{S}$, and prove that they are groupoids (hence the name 'groupoid fibration').

We show that quotient stacks (with appropriate projection functor into **Man**) are groupoid fibrations, and explain what the fibres are. Particularly, the projection $[X/G] \to$ **Man** sends an object $(P \to U, P \to X)$ to $U \in$ **Man**, and an arrow $(P \to Q, U \to V)$ to the arrow $U \to V \in$ **Man**.

*n.schaaf@student.ru.nl

Next, we explain what *restrictions* are in groupoid fibrations (pullbacks along inclusion maps), which are necessary for the definition of a stack. This includes restrictions of both objects and arrows.

Stacks. We state the definition of a ***stack*** (over **Man**) as a groupoid fibration over **Man** where objects and arrows can be *glued* (formal definition given in the lecture). We motivate the definition with illustrations, and relate the gluing properties to the transition functions on a manifold.

We claim that the manifold stacks $\underline{X}$ are stacks, and, depending on time, we may give a proof. More importantly, we will sketch a proof that quotient stacks $[X/G]$ are stacks. To do this, we describe what restrictions are in quotient stacks.

Rough timetable. The following is a rough estimate for the timeslots of each topic in the lecture, summarising the above description.

1. Recap [5 minutes].
2. Groupoid fibrations [10 minutes].
3. Pullbacks (are essentially unique) [5 minutes].
4. Fibres in groupoid fibrations [5 minutes].
5. Restrictions in groupoid fibrations (pullbacks along inclusion maps) [5 minutes].
6. Stacks [10 minutes].
7. Arguing quotient stacks are stacks [5 minutes].