Classical and Quantum Particles in Galilean and Poincaré Spacetime

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Outline

Symmetry in quantum mechanics

- Principle of relativity: laws of physics are invariant under Lorentz transformations.
- States are unit vectors $\psi, \varphi \in H$.
- Under Lorentz transformations $L \in \mathcal{P}$

$$\psi \xrightarrow{L} \psi'$$

transition probabilities are conserved:

$$|\langle \psi, \varphi \rangle|^2 = |\langle \psi', \varphi' \rangle|^2.$$

This is exactly what the principle of relativity implies.

Representations in quantum mechanics

- Eugene Wigner (1902–1995) proved that the transformation $\psi \xrightarrow{L} \psi'$ must be realised by a *unitary transformation* $D(L)$ on the Hilbert space H:

$$\psi' = D(L)\psi.$$

- Physically the states $D(L_2)D(L_1)\psi$ and $D(L_2L_1)\psi$ should be the same.
- Since ψ and $e^{i\theta}\psi$ represent the same physical state:

$$D(L_2)D(L_1) = \omega(L_2, L_1)D(L_2L_1),$$

where $\omega(L_2, L_1)$ is a complex phase.

- D is known as a **projective unitary representation**.

Elementary particles in quantum mechanics

- State space of a physical system is represented by a Hilbert space H.
- Elements $L \in G$ of symmetry group act on the Hilbert space via the representation $D(L)\psi$.
- Subspace $H_0 \subseteq H$ corresponding to elementary particle.
  - H_0 should be transformed into itself.
  - H_0 should not have smaller subspaces that are invariant under the symmetries.
- Therefore: elementary particles are defined as **irreducible** projective unitary representations.

Elementary particles in quantum mechanics

- Goal: classify the quantum elementary particles.
- When $\omega \equiv 1$, D becomes an ordinary (non-projective) unitary representation. These are easier to work with.
- Projective unitary representations are classified via *ordinary* unitary representations (unirreps) of **extensions** of the symmetry group.

Spacetime symmetry groups: the Galilei group

- Non-relativistic spacetime: $\mathbb{R} \times \mathbb{R}^3$.
- The **Galilei group** $\mathcal{G}$ characterises Newtonian coordinate transformations ($c = \infty$):

$$\begin{aligned} \text{spacetime translations:} \quad & (t, \mathbf{x}) \longmapsto (t + s, \mathbf{x} + \mathbf{a}) \\ \text{spatial rotations:} \quad & (t, \mathbf{x}) \longmapsto (t, R\mathbf{x}) \\ \text{Galilean boosts:} \quad & (t, \mathbf{x}) \longmapsto (t, \mathbf{x} + t\mathbf{v}). \end{aligned}$$

- Formally $\mathcal{G}$ is a *semi-direct product*:

$$\mathcal{G} = \mathbb{R}^4 \rtimes \mathrm{E}(3) = \underbrace{\mathbb{R}^4}_{\text{translations}} \rtimes \underbrace{(\mathbb{R}^3 \rtimes \mathrm{SO}(3))}_{\text{boosts and rotations}} .$$

Spacetime symmetry groups: the Poincaré group

- Special relativistic spacetime: Minkowski space $\mathbb{M}$.
- The **Poincaré group** $\mathcal{P}$ characterises (inhomogeneous) Lorentz transformations:

$$\begin{aligned} \text{spacetime translations:} \quad & (t, \mathbf{x}) \longmapsto (t+s, \mathbf{x}+\mathbf{a}) \\ \text{Lorentz transformations:} \quad & (t, \mathbf{x}) \longmapsto \Lambda(t, \mathbf{x}). \end{aligned}$$

- Formally $\mathcal{P}$ is also a semi-direct product:

$$\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L}.$$

Extensions and unirreps of the rotation group

- The rotation group $\mathrm{SO}(3)$ describes all rotations of three-dimensional Euclidean space.
- Its relevant extension is $\mathrm{SU}(2)$.
- The unirreps of $\mathrm{SU}(2)$ are well-known: they are labelled by a non-negative half-integer number $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ called *spin*. The corresponding Hilbert space is $H = \mathbb{C}^{2j+1}$.

Extension and unirreps of the Galilei group

- The extension of the Galilei group that we need is:

$$\check{\mathcal{G}} = \mathbb{R}^5 \rtimes (\mathbb{R}^3 \rtimes \mathrm{SU}(2)).$$

- The resulting unirreps are labelled by the three parameters:
  - The **mass** of the particle: $m \in \mathbb{R}$.
  - The **spin** of the particle: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
  - An additional parameter $V \in \mathbb{R}$, interpreted as **internal energy**.
- But: the internal energy V makes no difference for the *projective* representations, so the elementary particles are only labelled by mass m and spin j.

Quantum elementary particles in Galilean spacetime

- Elementary particles are labelled by mass m and spin j.
- We have two physically relevant cases:
  - $m > 0$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (massive particles)
  - $m = 0$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (massless particles)
- Tachyons ($m < 0$) are allowed in this formalism. When charge is taken into account, these can be interpreted as anti-particles.

Quantum elementary particles in Poincaré spacetime

- For the Poincaré group the relevant extension is:

$$\check{\mathcal{P}} = \mathbb{R}^4 \rtimes \mathrm{SL}(2, \mathbb{C}).$$

- The classification of the unirreps is as follows. It is sometimes known as *Wigner's classification*.
  - The **mass** m.
  - When $m > 0$ we also have a **spin** $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
  - When $m = 0$ we have instead a parameter h called the **helicity**. It can be a real number (*continuous spin*), or any half-integer: $h = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$
- The continuous spin case does not occur in nature.

Naturally occurring particles

- Actual elementary particles are classified by projective unirreps of the Poincaré group $\mathcal{P}$ (the Galilei group is only an *approximate* symmetry).
- The naturally occurring representations are as follows:
  - $m > 0$, $j = \frac{1}{2}$: **leptons** and **quarks**.
  - $m > 0$, $j = 0$: **Higgs boson**.
  - $m > 0$, $j = 1$: **electroweak bosons**.
  - $m = 0$, $h = \pm 1$: **photons** and **gluons**.
  - $m = 0$, $h = \pm 2$: **gravitons**.

Conclusion and outlook

- We have a formalism that predicts the existence of elementary particles with familiar properties.
- The formalism does not tell us why only certain values of mass and spin occur in nature.
- What is the exact relation between the classical and quantum formalism?