Causal Coverage

LRB26 · Budapest

Nesta van der Schaaf

jww Chris Heunen

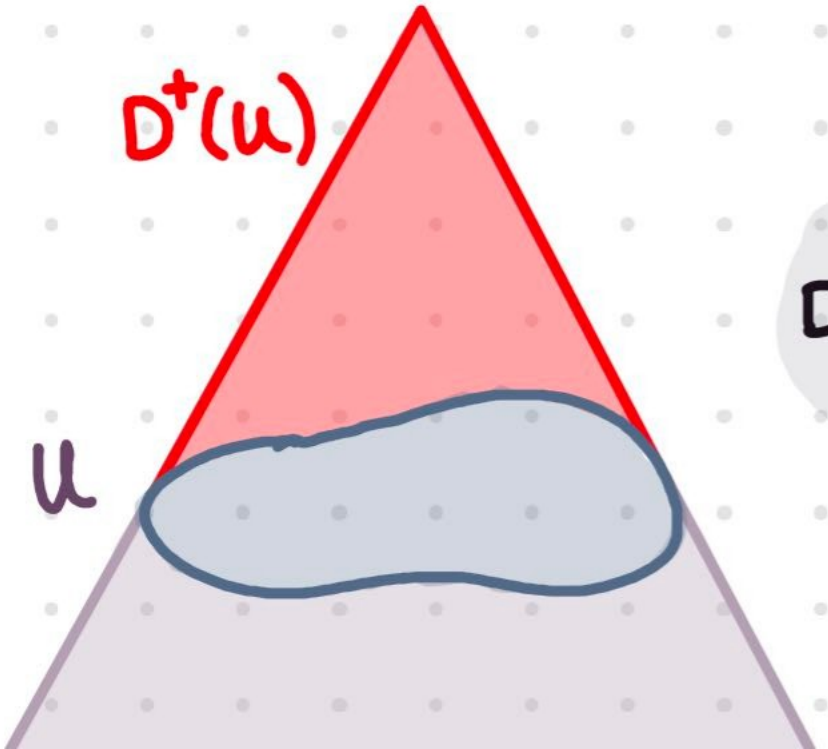
15 *July* 2026

arXiv:2510.17417
To appear in JMP

Inria

Causal Coverage

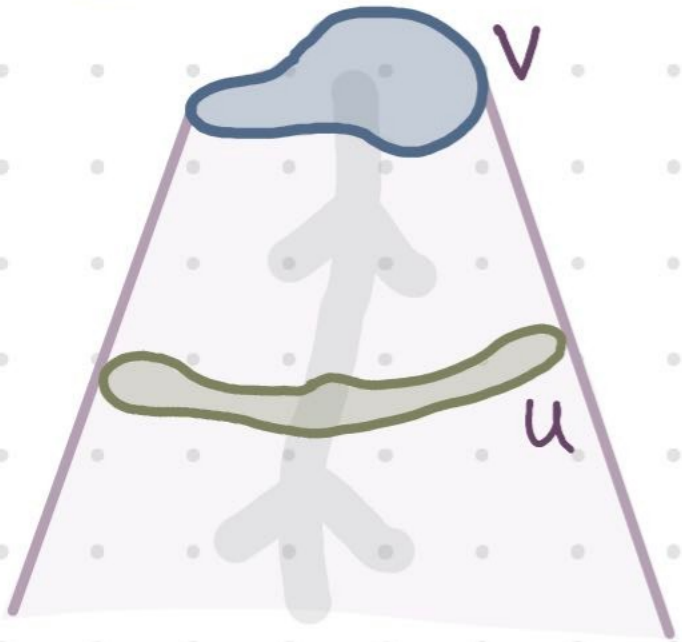
[Geroch 70]



$$D^+_c(A) := \left\{ x \in M : \begin{array}{l} \text{every past inextb.} \\ \text{causal curve through} \\ x \text{ intersects } A \end{array} \right\}$$

Causal Coverage

[Christensen, Crane 05]



IDEA all info. reaching V must pass through U:

$$U \in \overline{Cov}(V)$$

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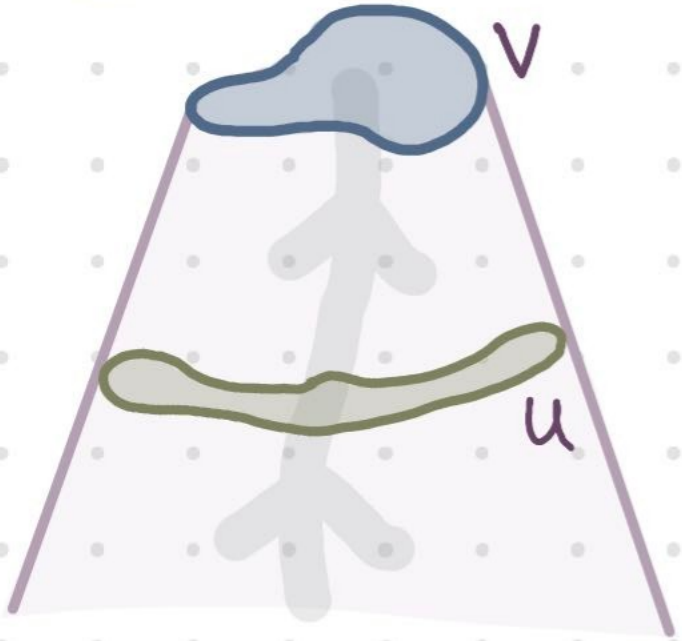

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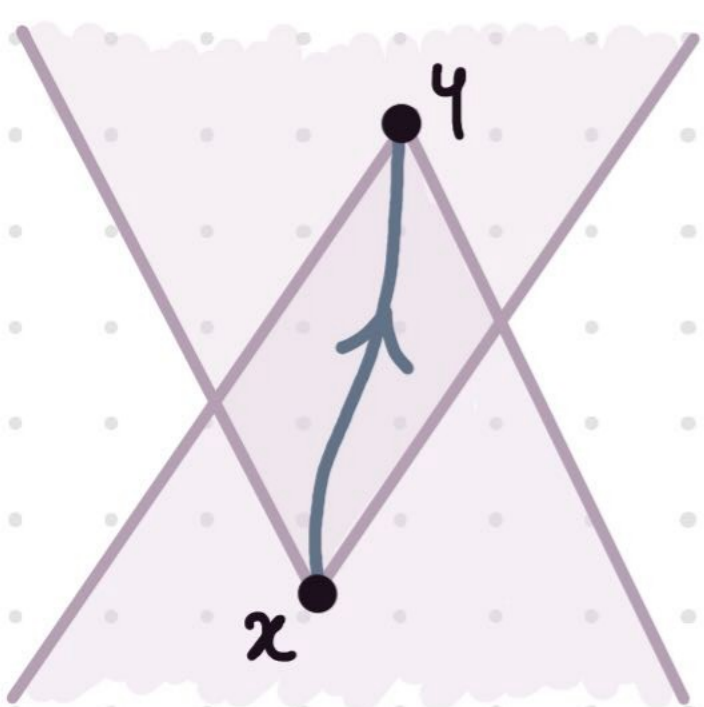
Goal:

causality theory

⇕

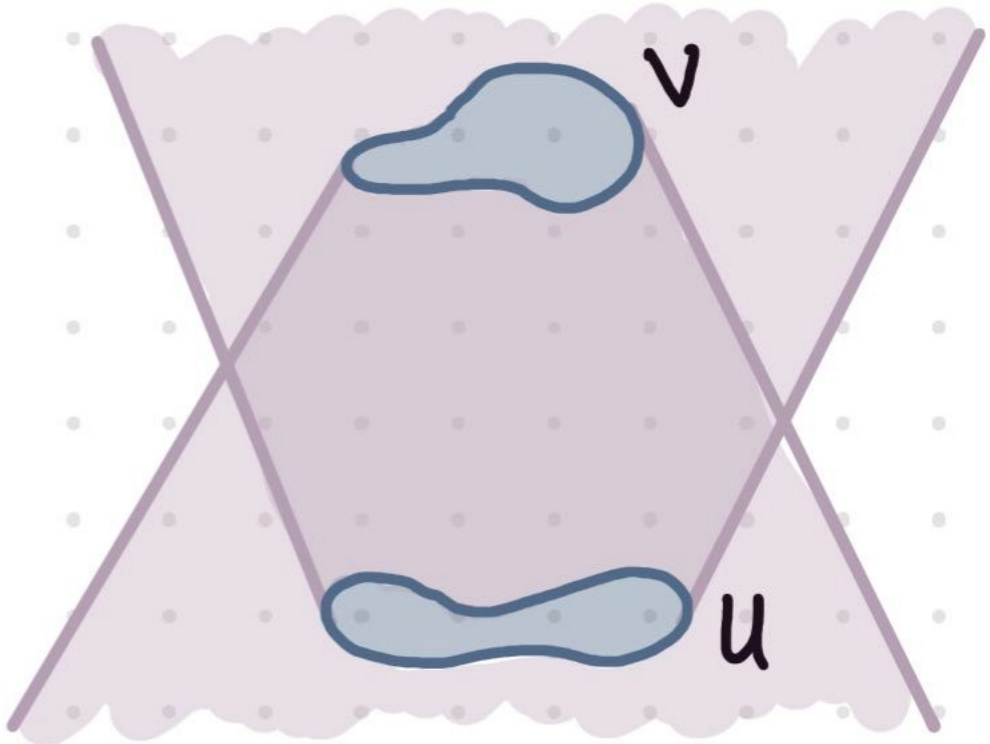
sheaf theory

Idea



causal order

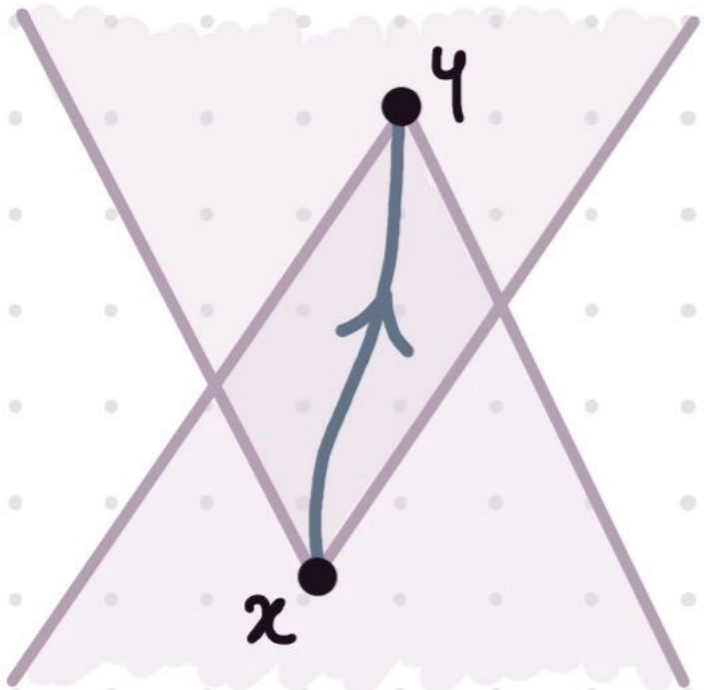
$x \leqslant y$



region causality

$U \trianglelefteqslant V$

Idea



Causal order

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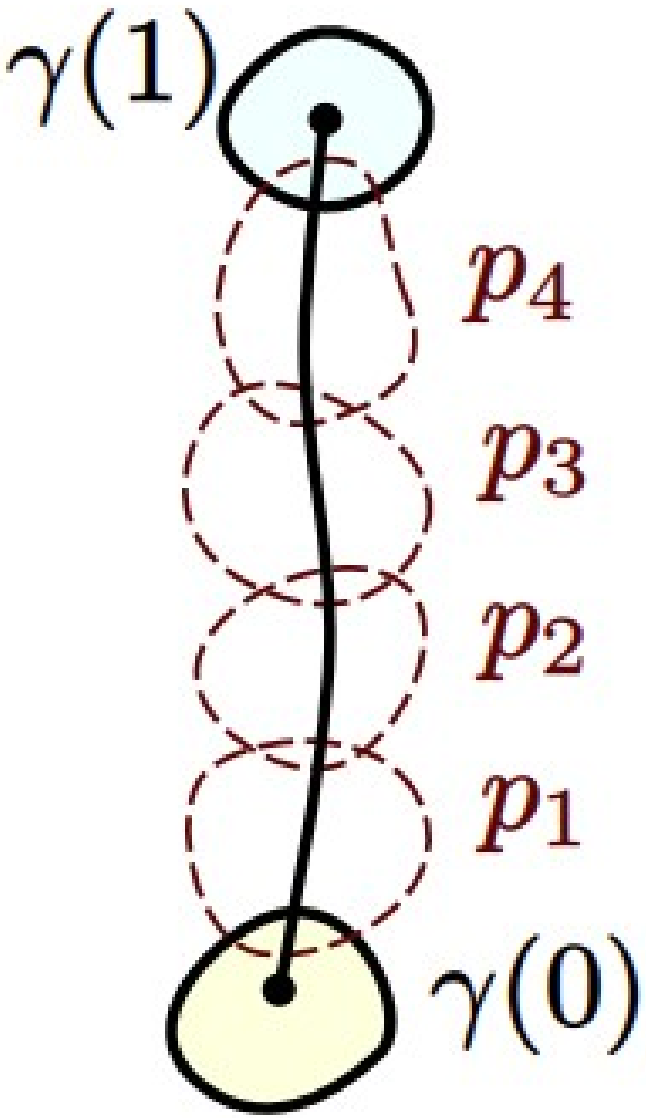

"the most important feature of space is not the points which it contains but rather the open subsets and the lattice relations between them. But then, since, physically speaking, a 'point' is a most peculiar concept anyway, why not drop it altogether and deal directly with frames/locales?"

C. J. Isham. "An Introduction to General Topology and Quantum Topology". In: *Physics, Geometry and Topology*. Ed. by H. C. Lee. NATO ASI Series. Boston, MA: Springer US, 1990, pp. 129–189 (cit. on p. 1).

Causal Coverage

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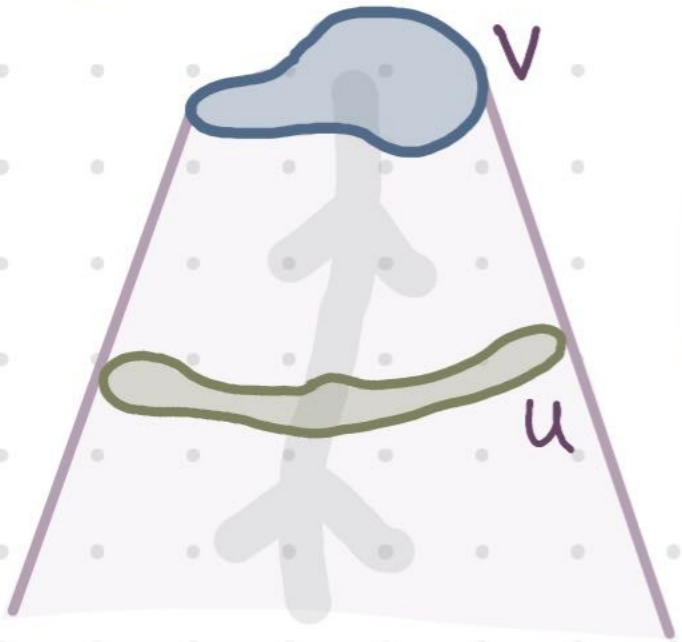
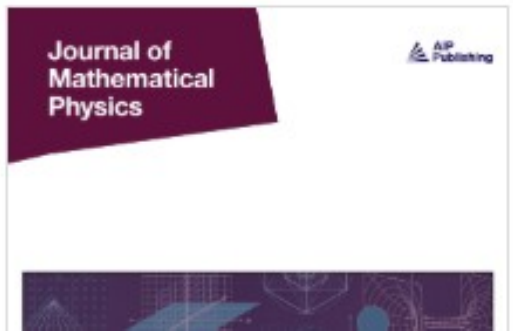


Volume 46, Issue 12
December 2005



RESEARCH ARTICLE | DECEMBER 15 2005

Causal sites as quantum geometry

J. Daniel Christensen; Louis Crane

+ Author & Article Information
J. Math. Phys. 46, 122502 (2005)
https://doi.org/10.1063/1.2138043 **Article history**

IDEA all info. reaching V must pass through U:

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Sheaf theory?

Sheaf theory?

- **Sheaves:** local to global
- **Lattice theory:** generating frames
- **Algebraic geometry**: schemes, étale cohomology
- **Logic**: topos theory, forcing
- **Constructive math**: formal topology (Sambin)

Sheaf theory?

- **Sheaves:** local to global
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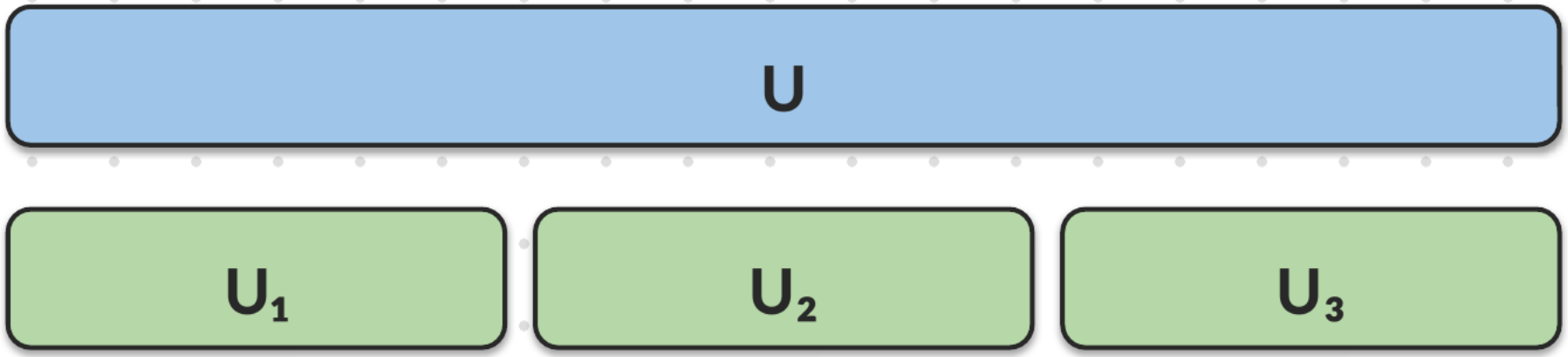
Sheaves:

local data glues to global data

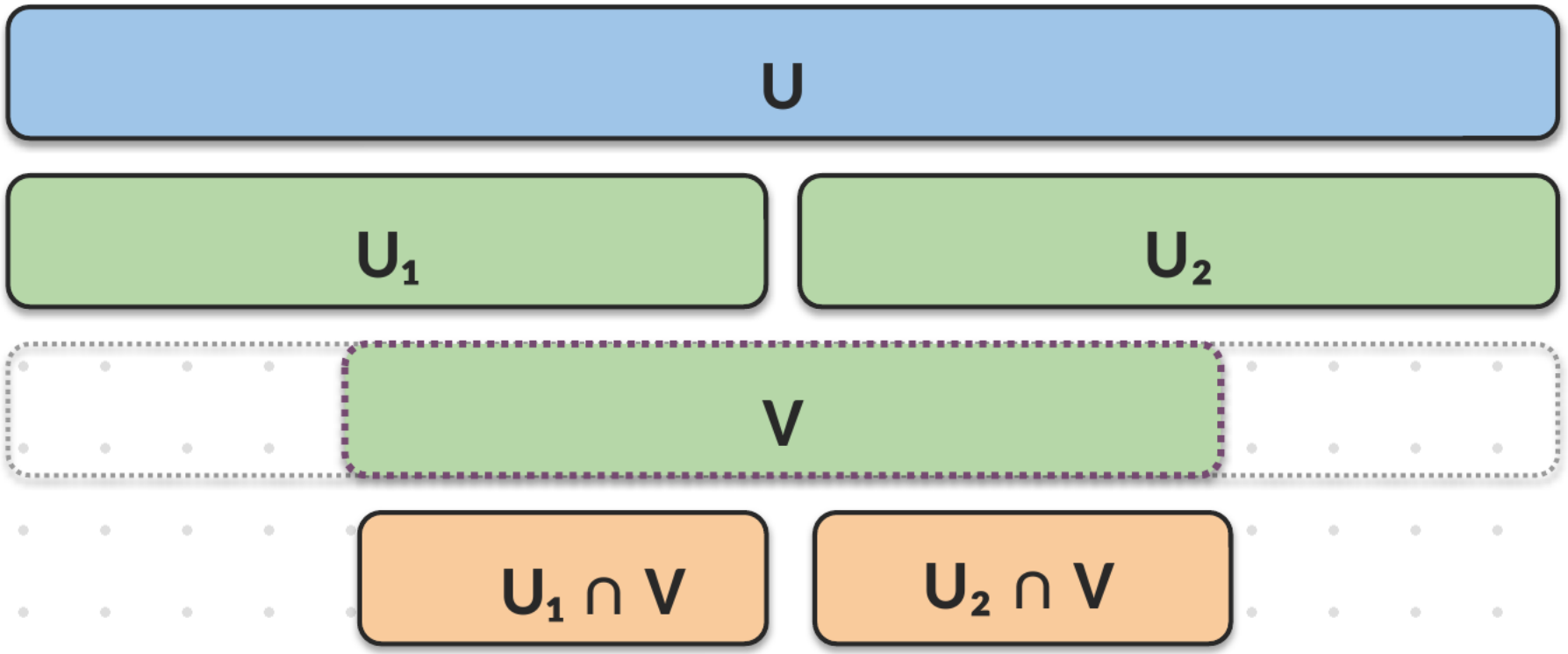
Coverage axioms

1. **Identity:** everything covers itself
2. **Stability:** covering is preserved under intersection
3. **Locality:** covering is local

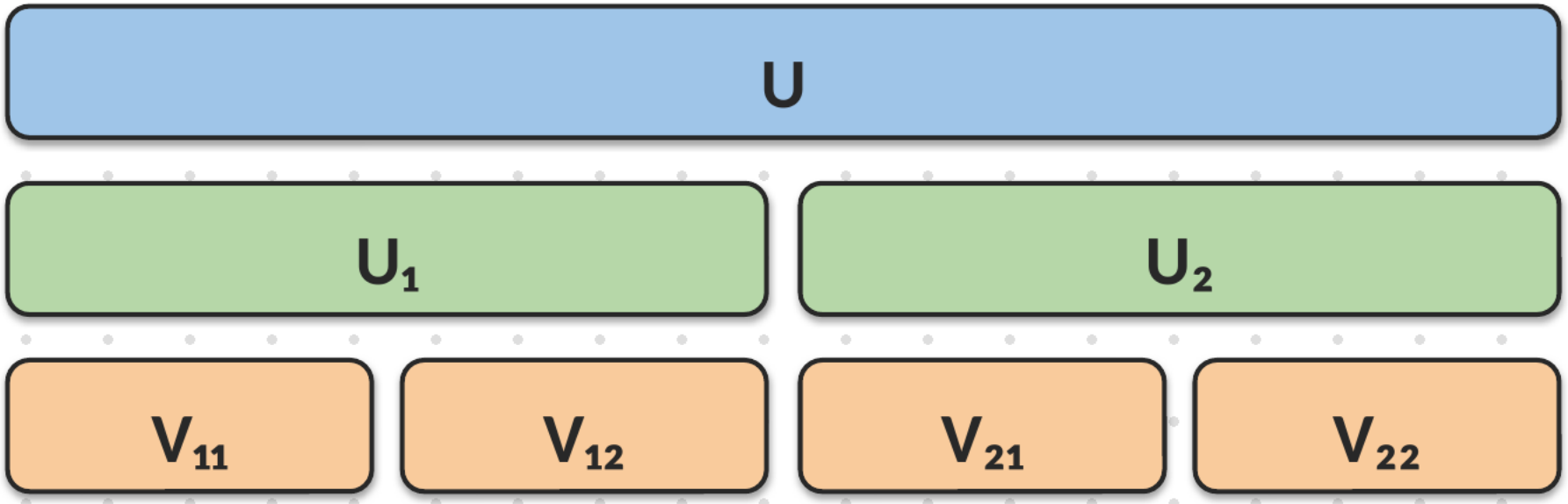
Basic intuition



Intersection stability

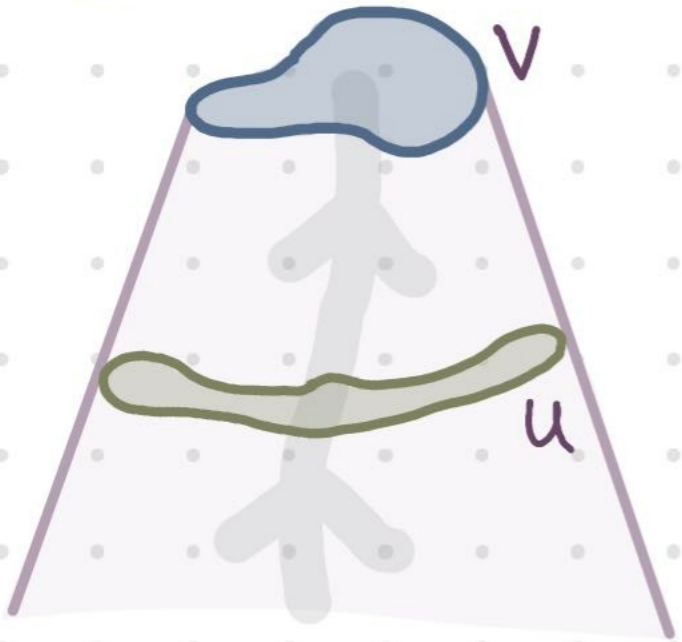


Local characteristic



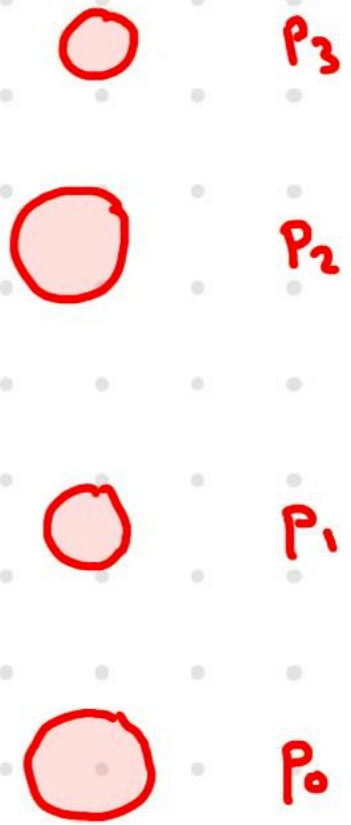
Causal Coverage

[Christensen, Crane 05]



PATHS a path is a finite chain:

$$P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_{N-1} \trianglelefteq P_N$$



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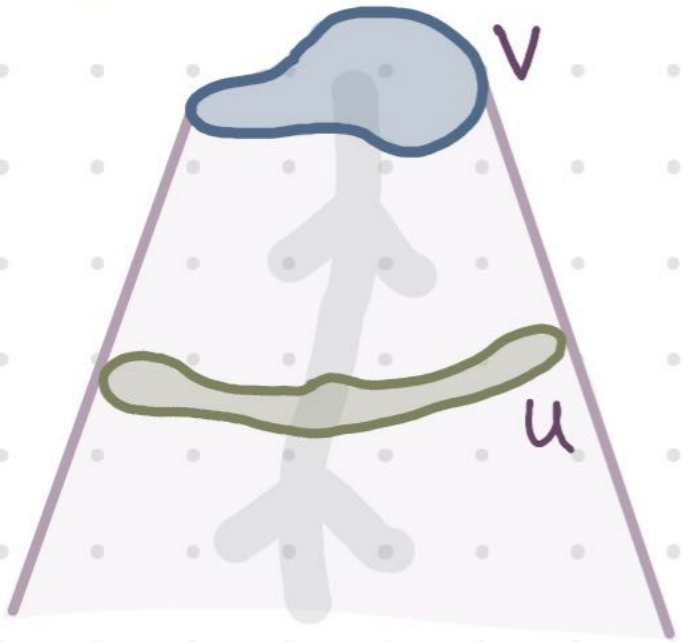

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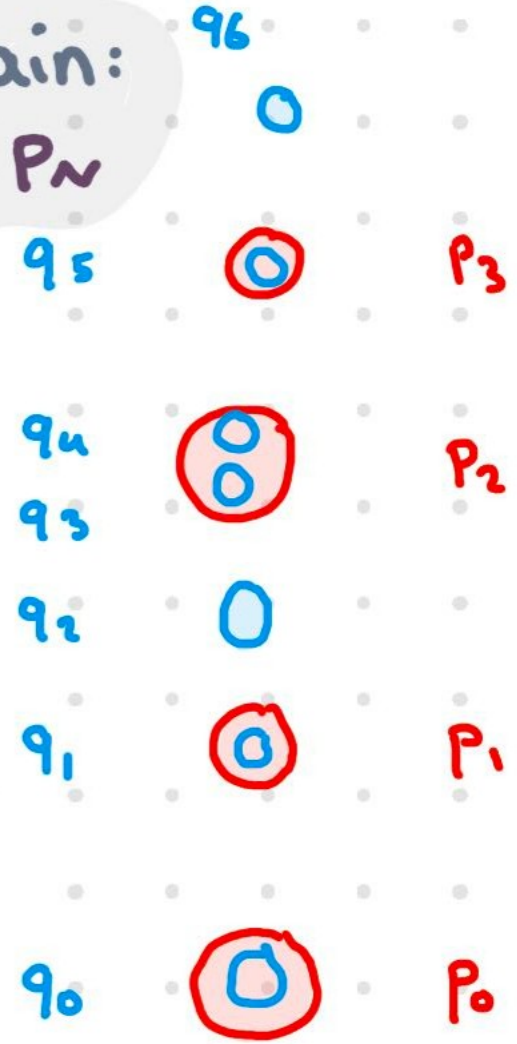
REFINES q refines p if:

$$\forall n \, \exists m : q_m \sqsubseteq p_n.$$



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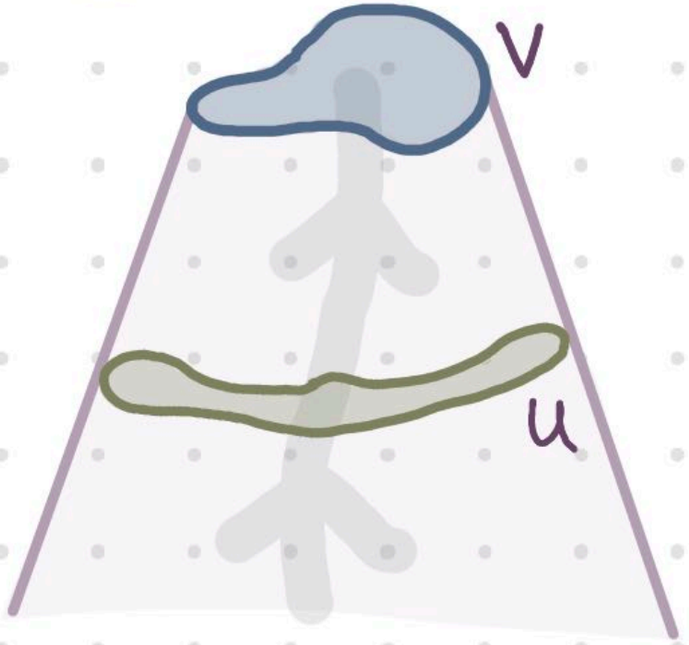

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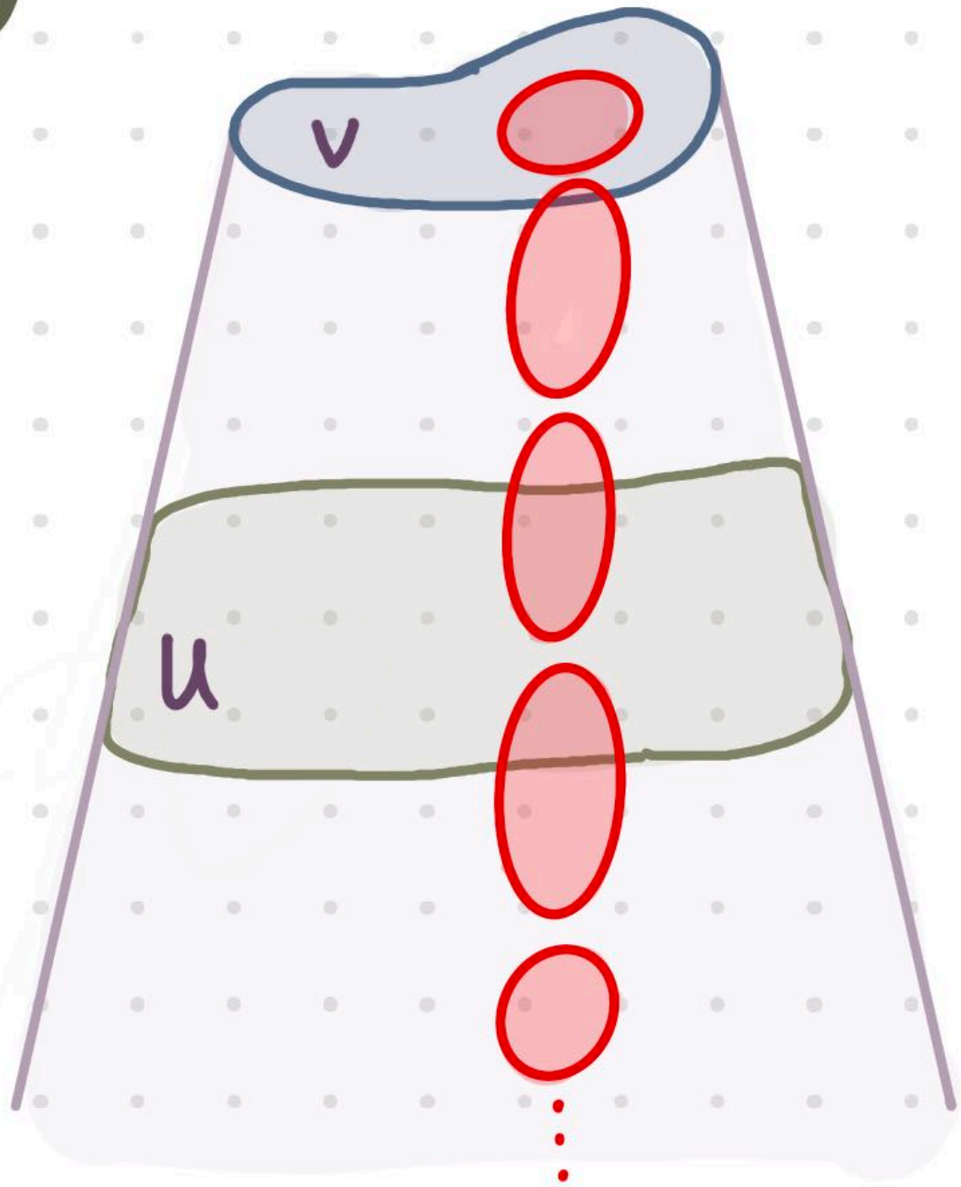

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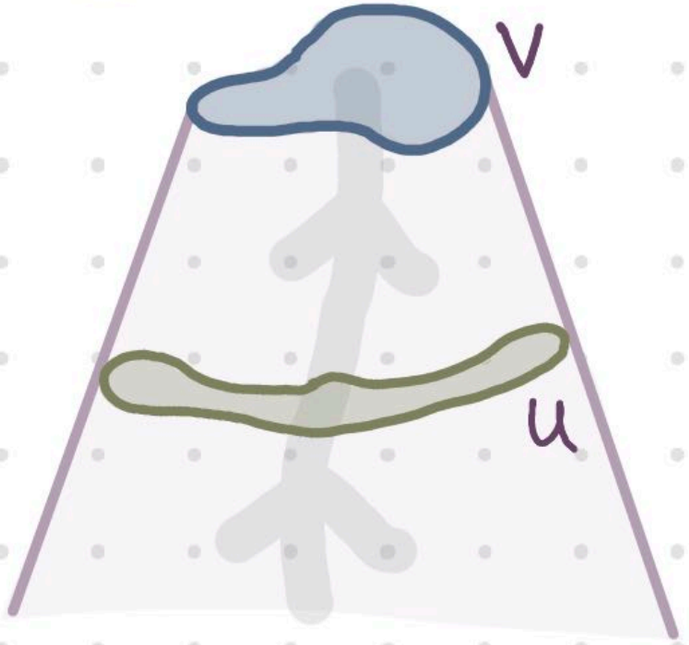

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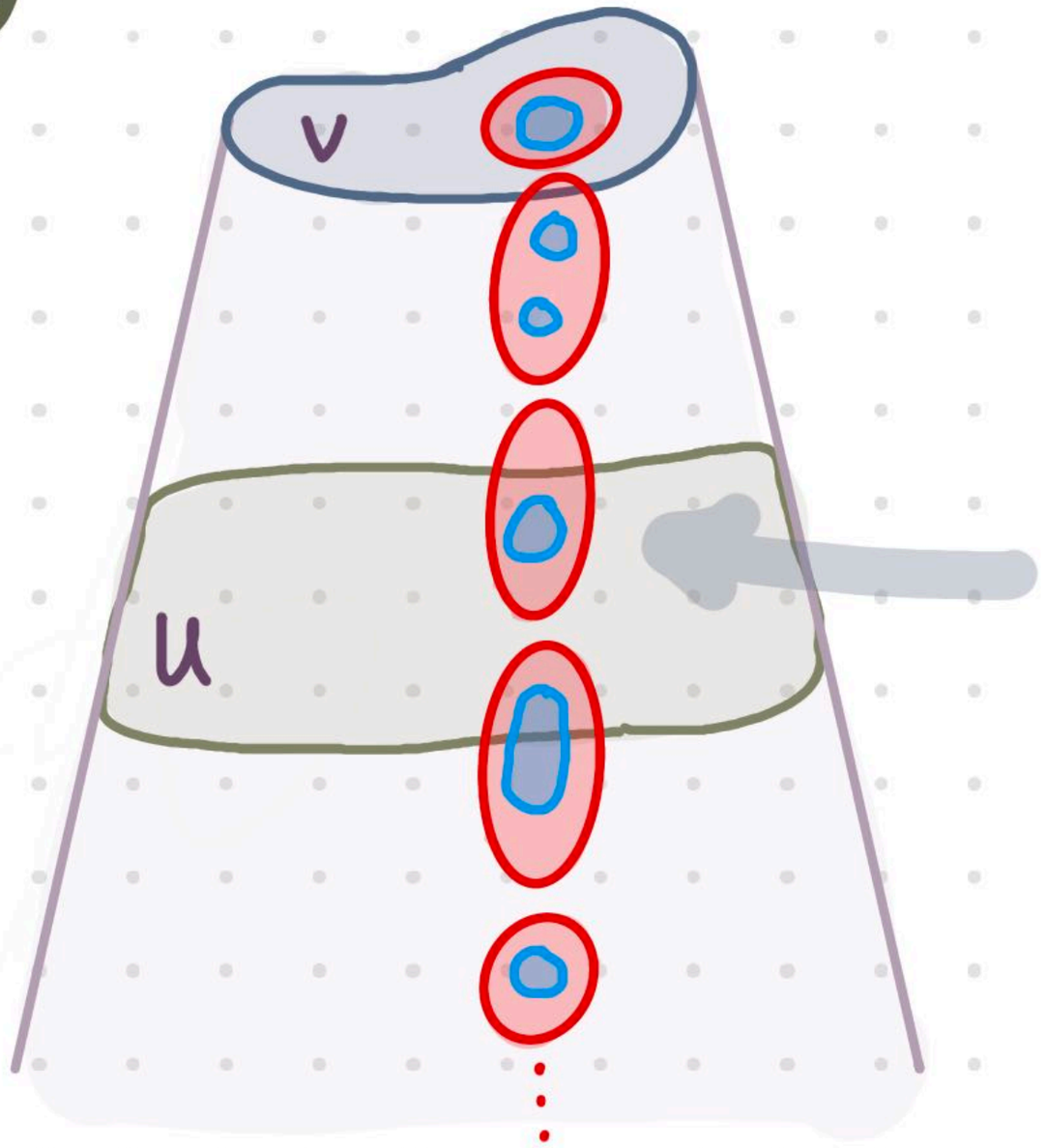

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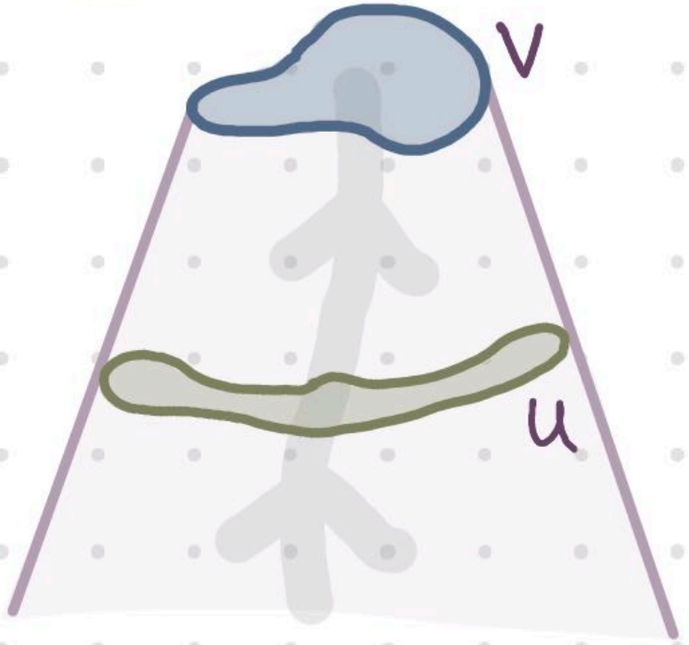

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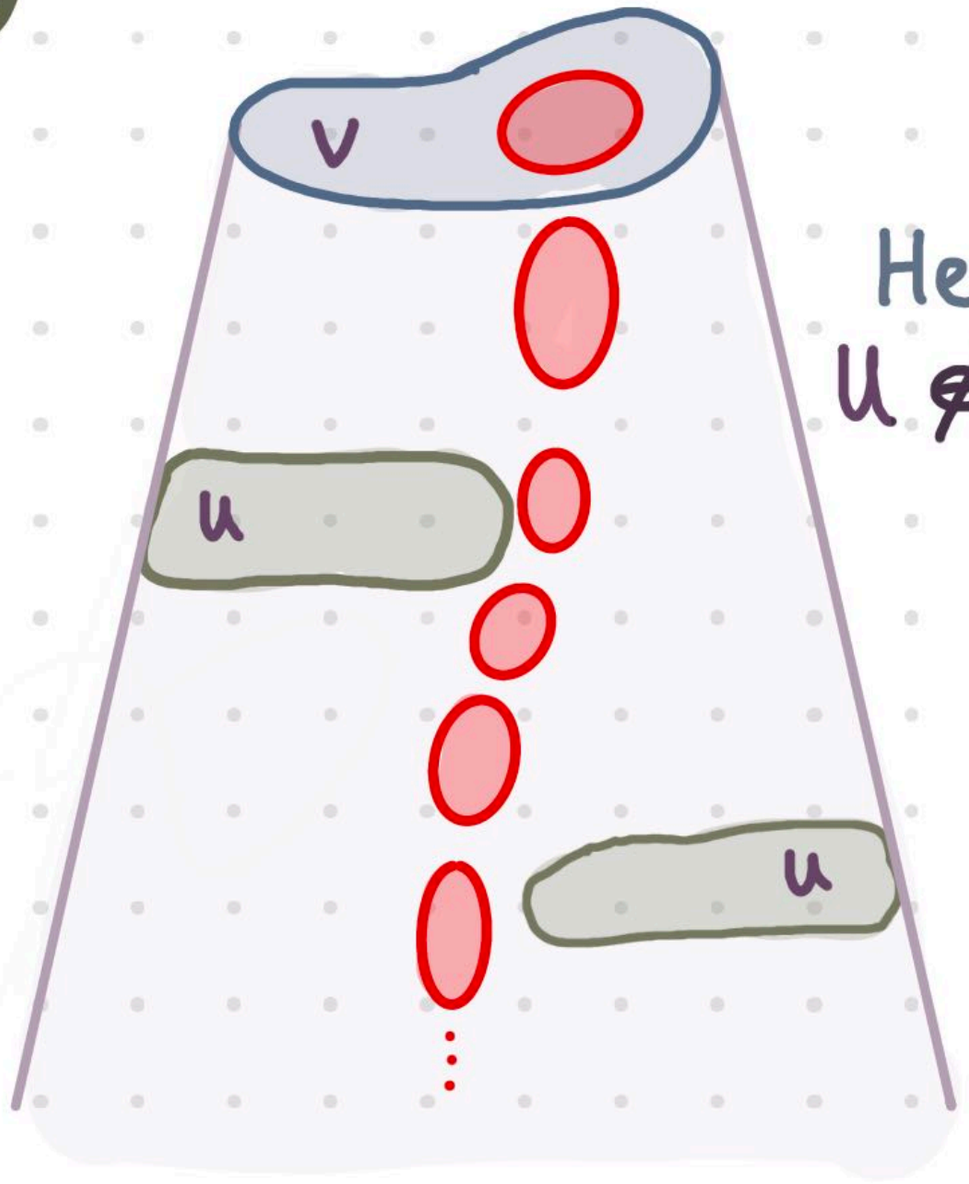

Here:

$$U \notin \overline{Cov}(V)$$

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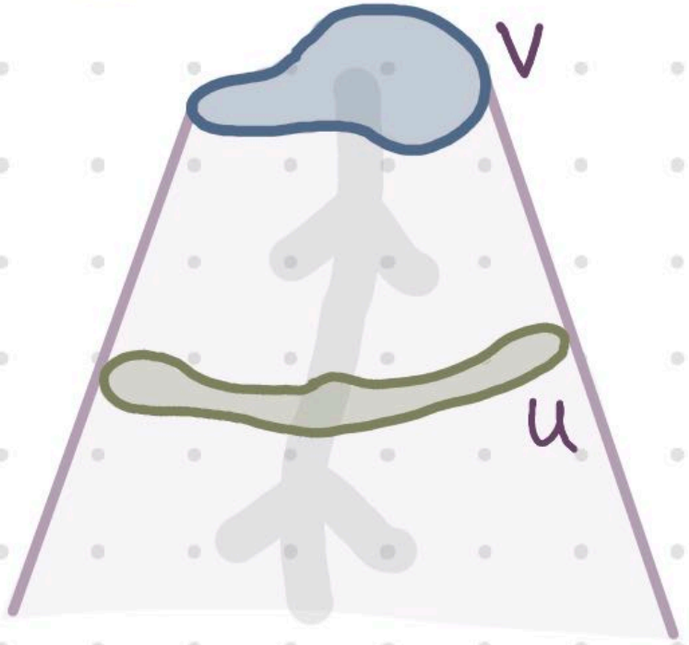

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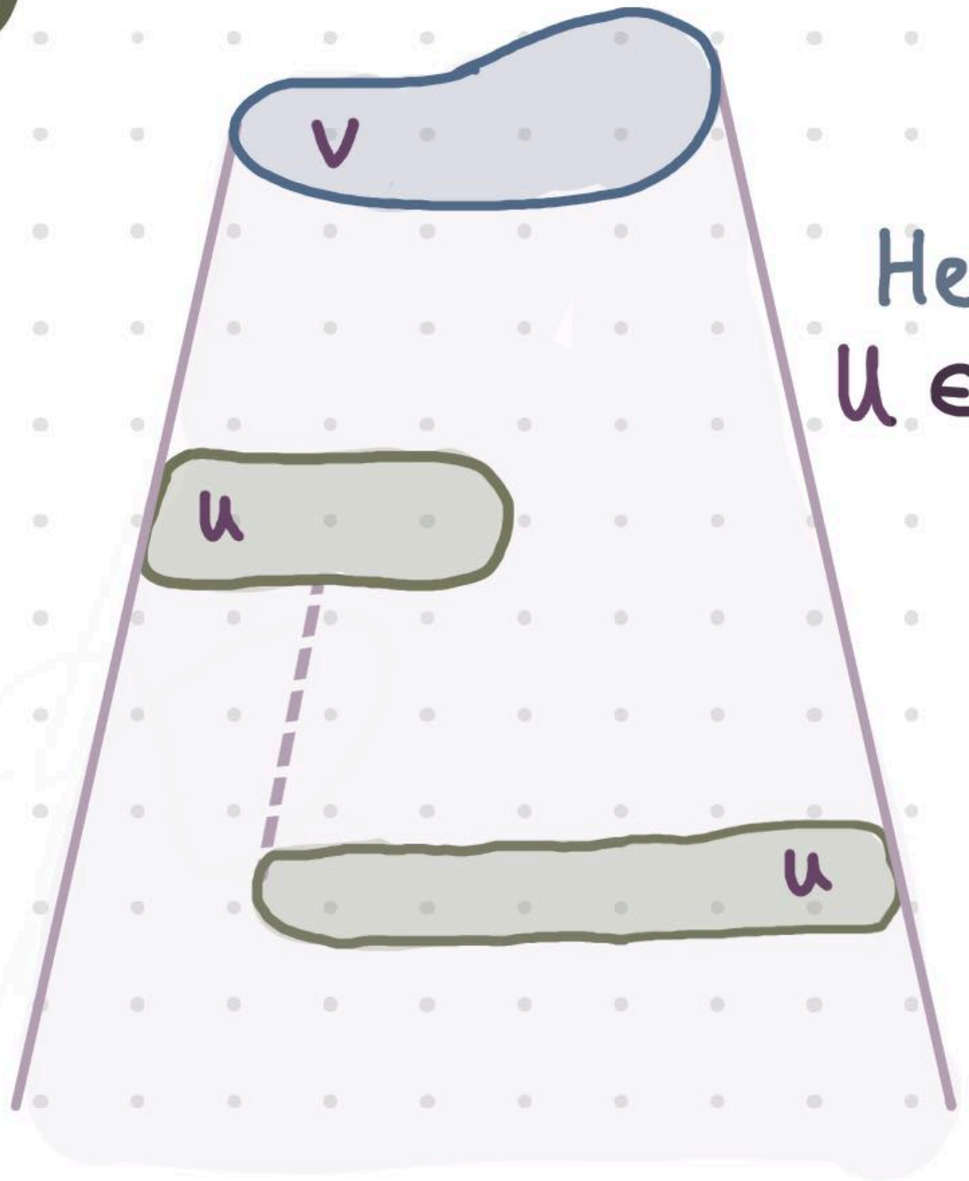

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Causal Coverage

LEMMA

- $U \in Cov^{-}(U)$
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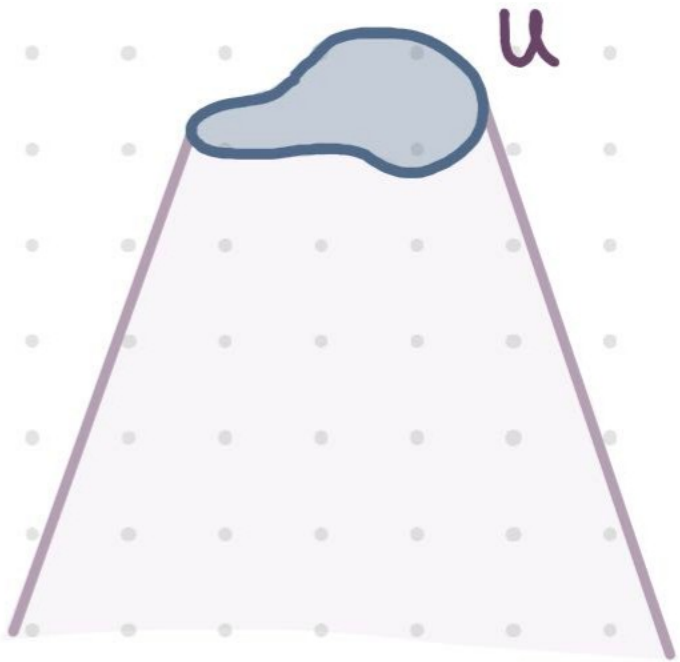

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- $U \in Cov^{-}(U)$
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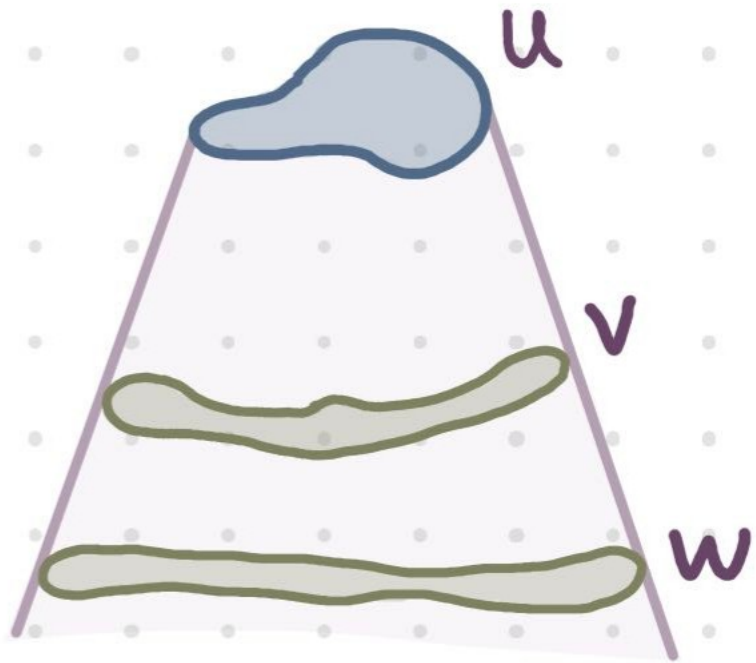

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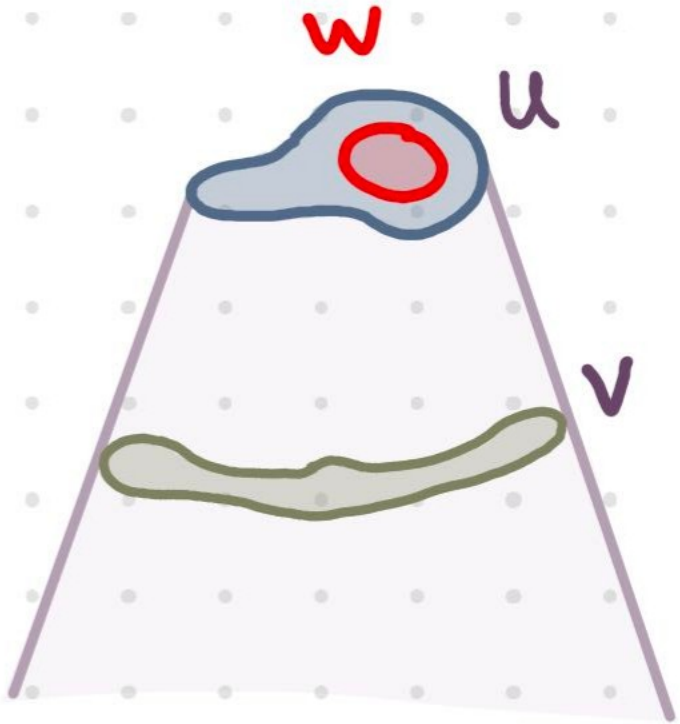
- $W \in \mathrm{Cov}^{-}(V),\ V \in \mathrm{Cov}^{-}(U)$
  $\Longrightarrow W \in \mathrm{Cov}^{-}(U)$
- $V \in \mathrm{Cov}^{-}(U),\ W \subseteq U$
  $\Longrightarrow V \wedge \downarrow W \in \mathrm{Cov}^{-}(W)$



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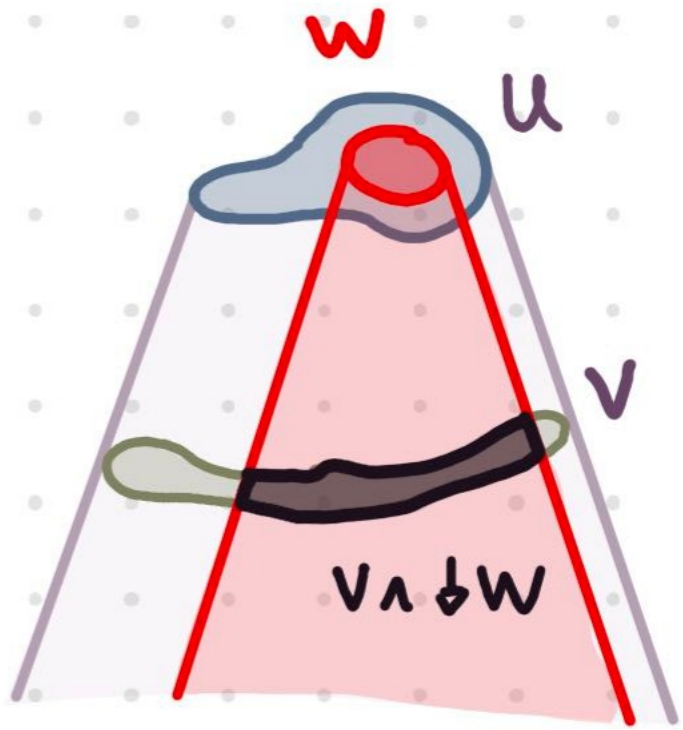

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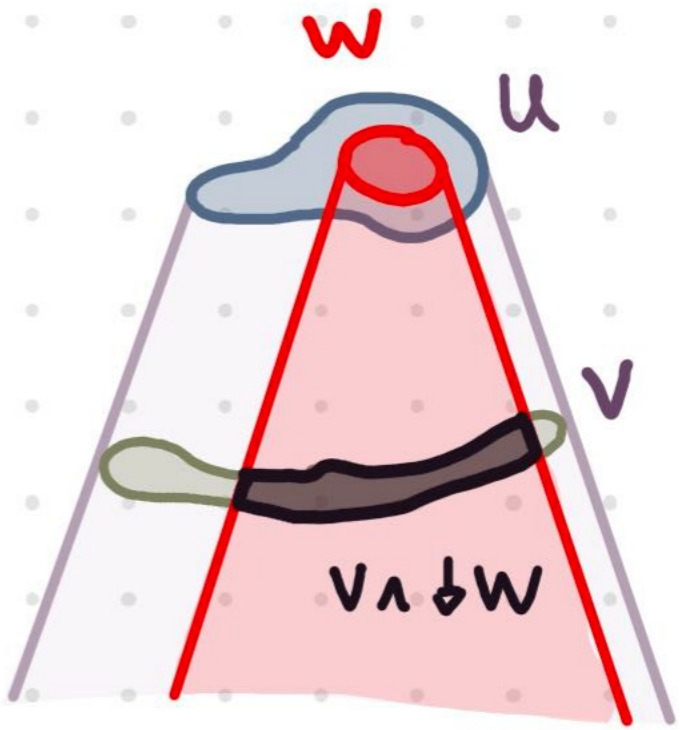

does not hold in
[Christensen Crane 05]

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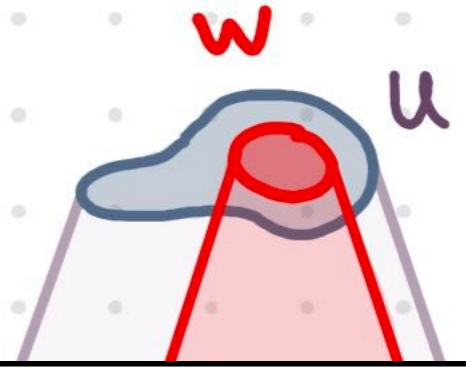


Remark 2.13.

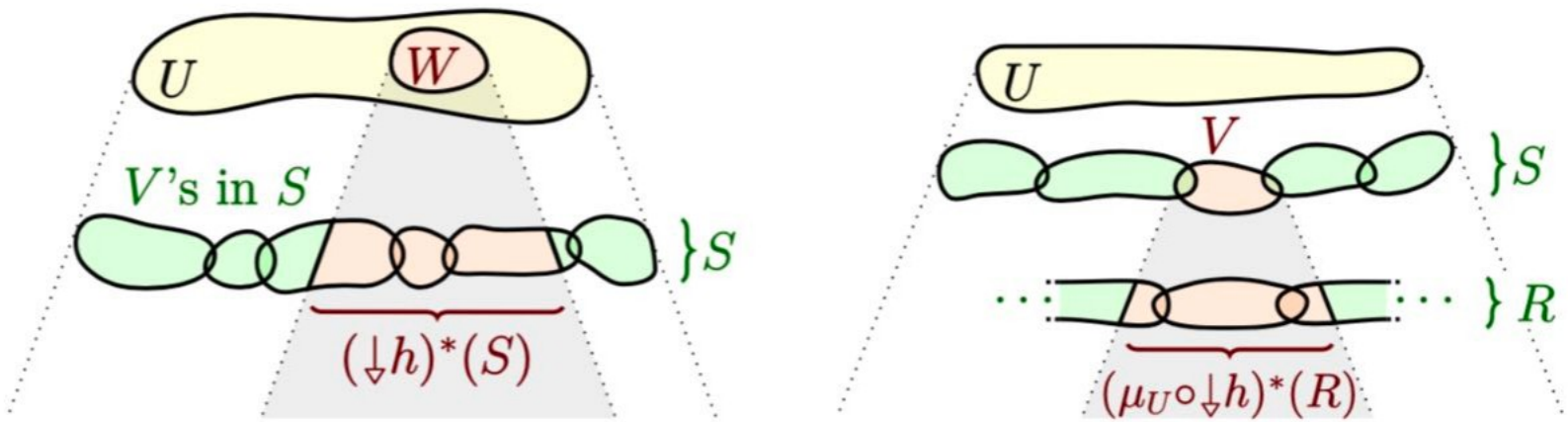
The collection of covers defined above has the feel of being a Grothendieck topology [Mac Lane et al 1992]. But even if pullbacks exist in this category (they do, for example, in the case of Example 2.1), the pullback of a cover is not necessarily a cover.

does not hold in [Christensen Crane 05]

Causal Coverage

DEF./THM.

For any ordered locale:
$\mathbf{Cou}^-$ naturally determines
a "$\downarrow$-Grothendieck top".



(a) Axiom (ii). (b) Axiom (iii).

Figure 24: Illustrations of the axioms of a $\downarrow$-Grothendieck topology for J^-.

Definition 6.4. Let $\mathbf{C}$ be a category, together with an endofunctor $T\colon \mathbf{C} \to \mathbf{C}$ and a natural transformation $\mu\colon T^2 \to T$. A (T, μ)-*Grothendieck topology* on $\mathbf{C}$ is a function J_T that assigns to each object $C \in \mathbf{C}$ a collection $J_T(C)$ of sieves on $T(C)$, called *covering sieves*, such that the following axioms are satisfied:

(i) the maximal sieve $t_{T(C)}$ is in $J_T(C)$;
(ii) if $S \in J_T(C)$ then $T(h)^*(S) \in J_T(D)$ for any arrow $h\colon D \to C$;
(iii) if $S \in J_T(C)$ and R is a sieve on $T(C)$ such that $(\mu_C \circ T(h))^*(R) \in J_T(D)$ for all $h\colon D \to T(C)$ in S, then $R \in J_T(C)$.

Takeaway:

Classical sheaves: *local* to *global*.

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Causal sheaves: *past* to *future*.

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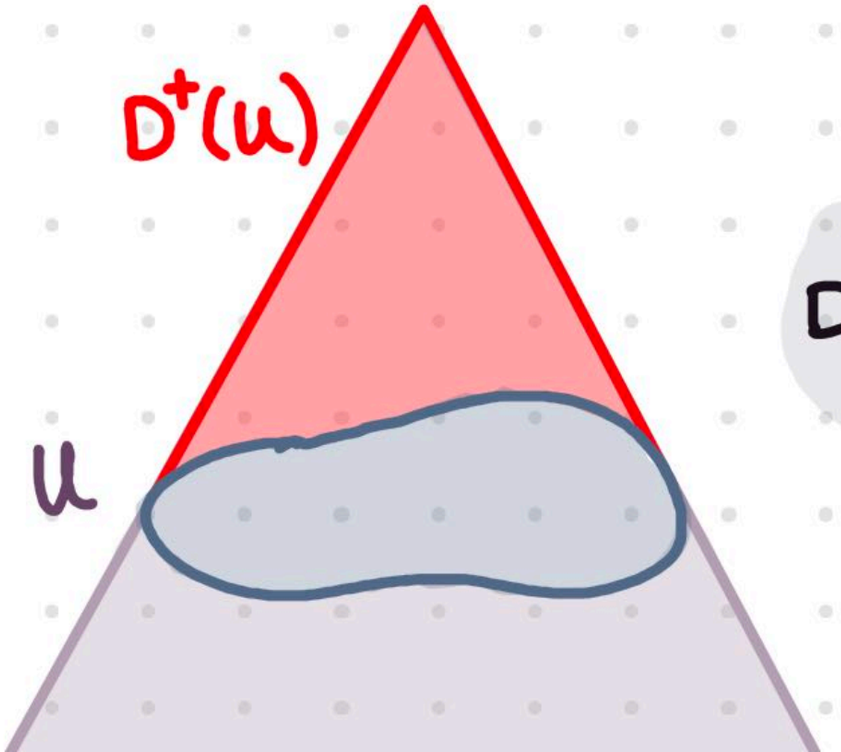
Causal sheaves: *past* to *future*.

Archetypal example: solutions to wave equation on Minkowski space

Causal Coverage

[Geroch 70]



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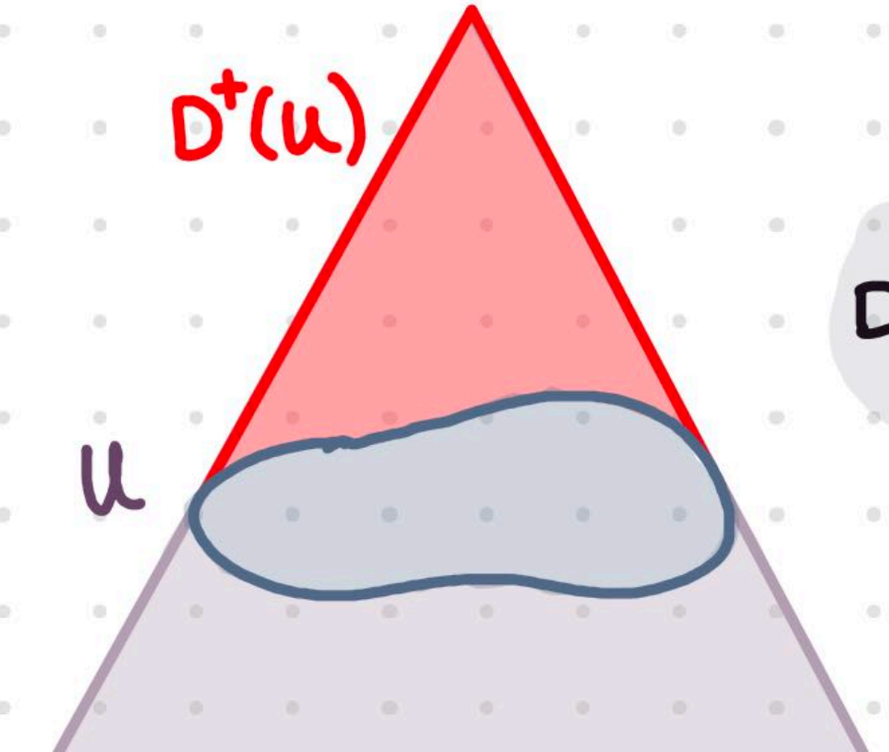
Causal Coverage

[Geroch 70]

DOMAIN OF DEPENDENCE

$$D^+(U) := \bigvee \{ W \in \sigma X : U \in \mathrm{Cov}^-(W) \}$$



$$D_c^+(A) := \left\{ x \in M : \begin{array}{l} \text{every past inextb.} \\ \text{causal curve through} \\ x \text{ intersects } A \end{array} \right\}$$

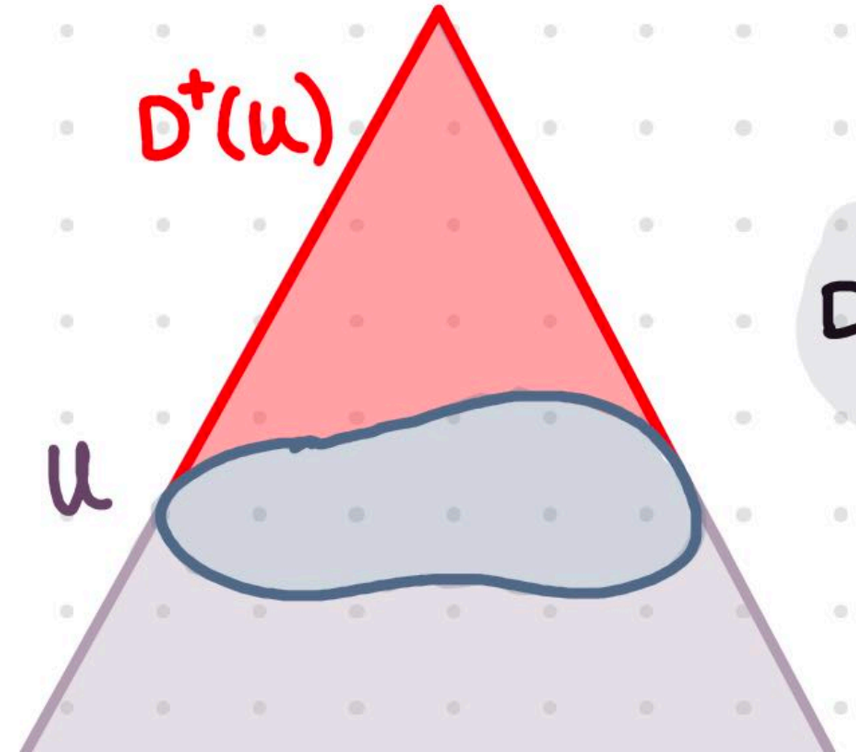
Causal Coverage

[Geroch 70]

DOMAIN OF DEPENDENCE

$$D^+(U) := \bigvee \{ W \in \sigma X : U \in Cov^-(W) \}$$

PROP. $D_c^+ \subseteq D^+$



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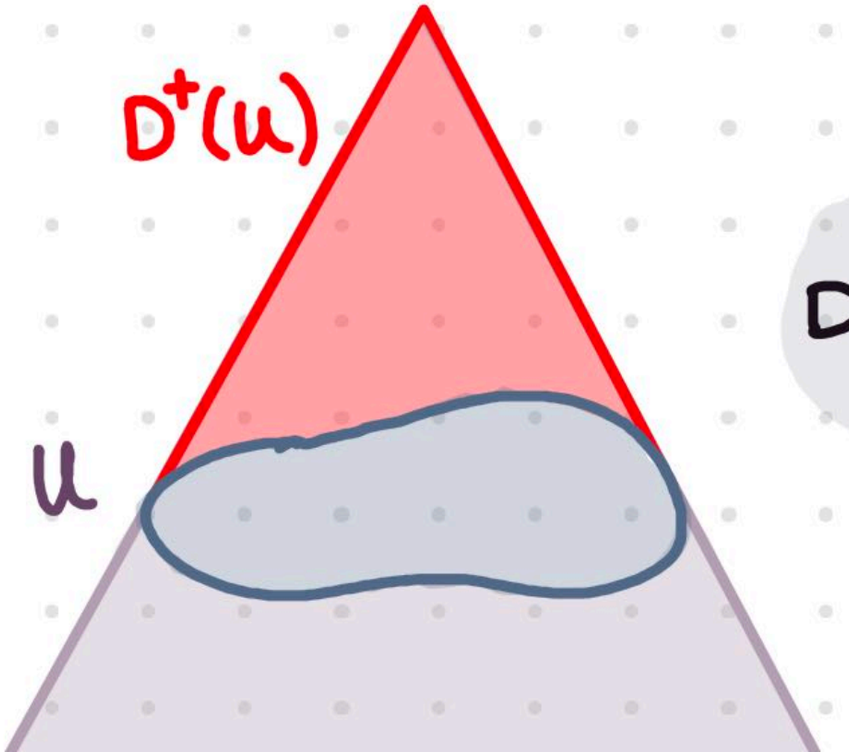
PROP. $D_c^+ \subsetneq D^+$



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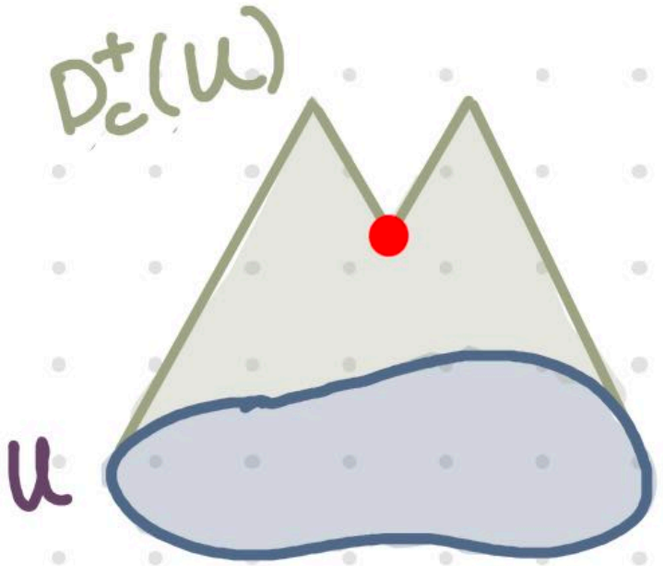
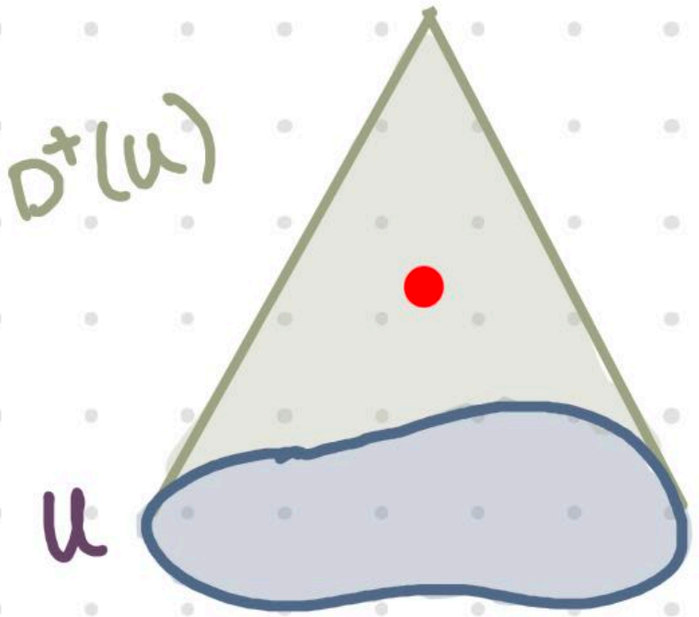


localic

curve-wise

Causal Coverage



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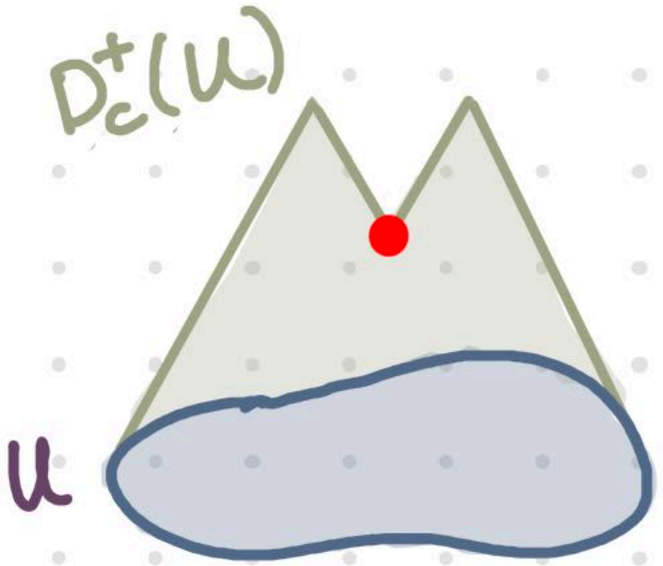

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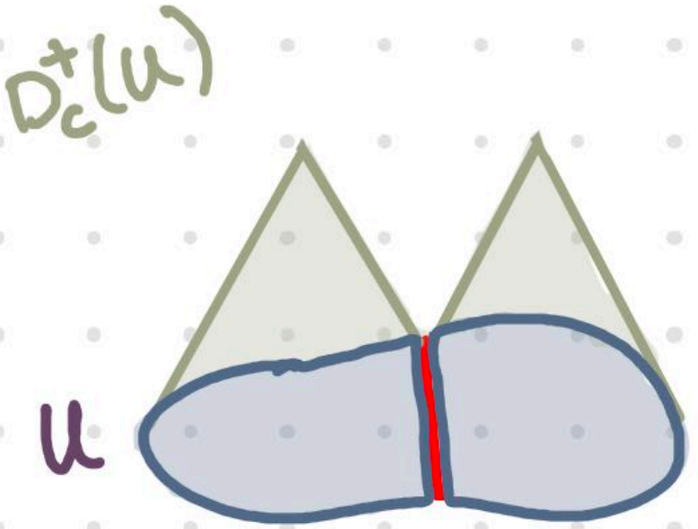

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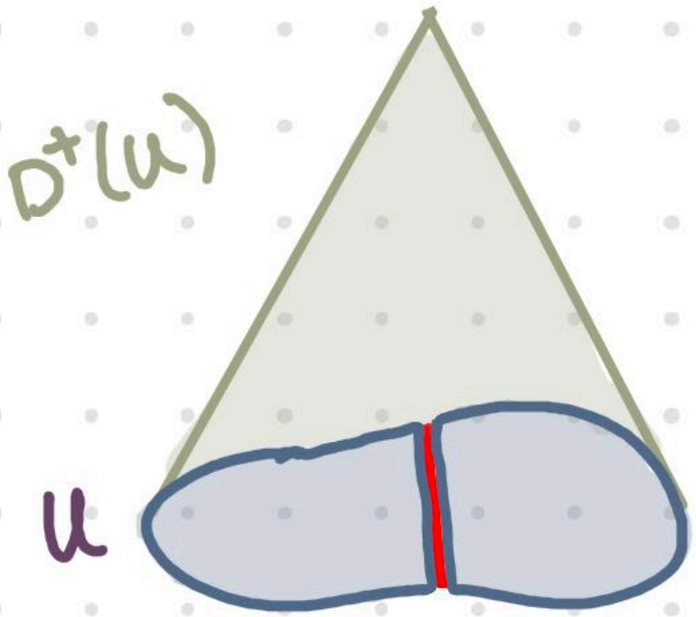

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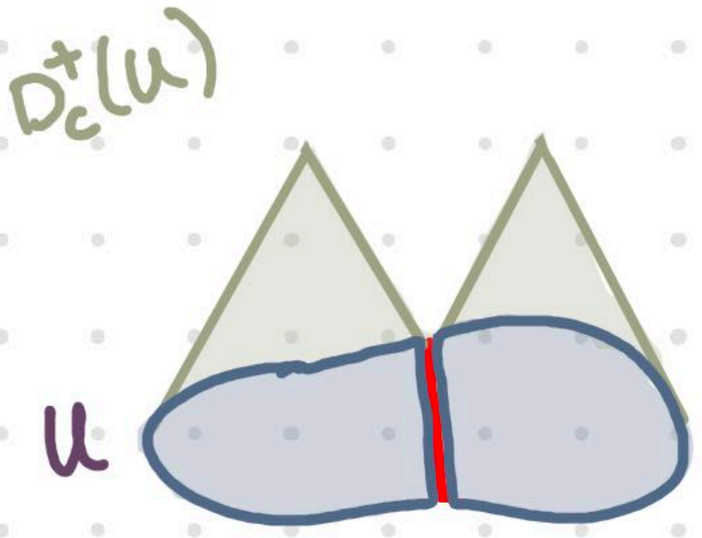
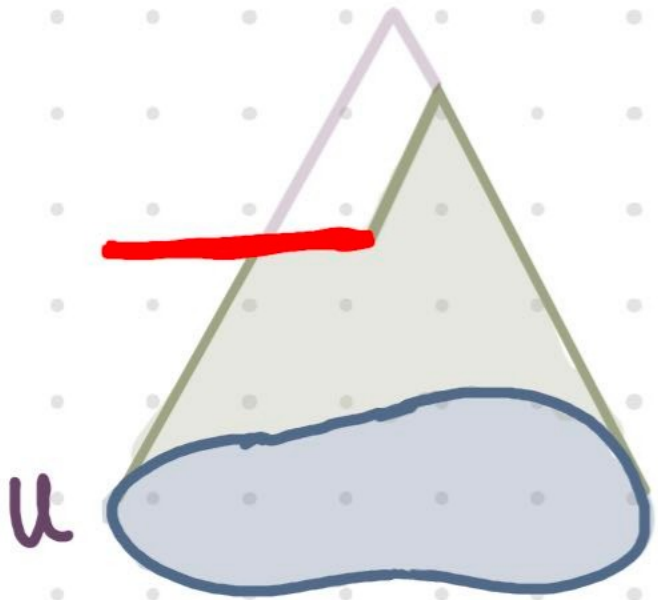


curve-wise

Causal Coverage



Q. Problem of holes?

"Even the Minkowski space is holed"
[Krasnikov 09]

sheaves?

EXAMPLE

Sheaf of solutions to the wave equation on Minkowski space