Some Results on Causal Modalities in General Spacetimes

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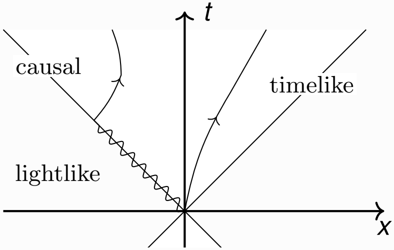
This suggest we can use:

– $\Diamond, \Box$ to study causal structure of spacetimes

– spacetimes to characterise certain modal logics

Causal Relations

– Different causal relations lead to different modalities;



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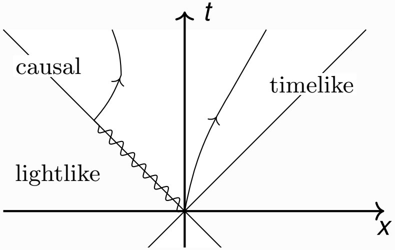

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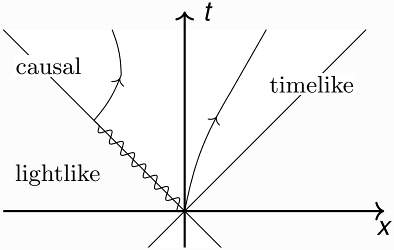

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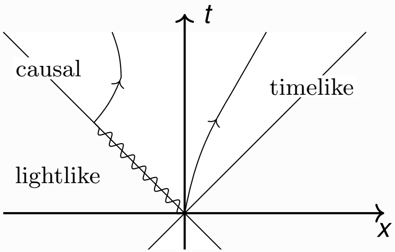

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- $x \; \alpha \; y$ iff there exists a causal curve from x to y;

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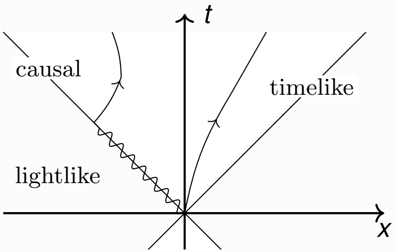

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- Not much known about modal logic of <u>general</u> spacetimes
- Other modalities, such as α, are not characterised yet

After Modality

The after relation

- α is the strict causal relation, in between $\ll$ and $\preccurlyeq$;
- Used by Robb in 1914 to axiomatise SRT;
- Goldblatt suggested that α might yield a more expressive modal logic;
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The 'after formula'

- α has basic properties such as transitivity;
- But loses an important one shared by $\ll, \preccurlyeq$; (2-density)
- $\mathsf{a}\alpha\mathsf{f}$ recaptures some of this lost property;
- Its validity on all spacetimes is a step toward characterising the after logic.

Kripke Semantics

$$\mathcal{F} = \langle W, R \rangle, \qquad \mathcal{M} = \langle \mathcal{F}, V \rangle$$

– W: set of possible worlds;

– $R \subseteq W \times W$: accessibility relation;

– $V(p) \subseteq W$: worlds where p is true.

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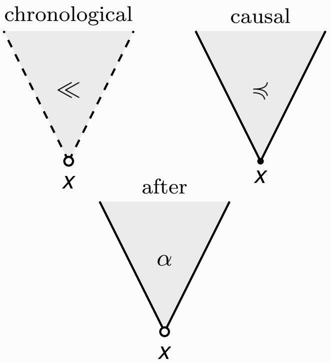
The modal logic of a frame collects all such formulas that are satisfied for all valuations:

$$\mathcal{L}\langle W, R \rangle := \{\varphi : \langle W, R \rangle \models \varphi\}.$$

Kripke Semantics

For a spacetime (M, g):

- W is the manifold M
- $R = \ll, \preccurlyeq, \alpha, \rightarrow, \underline{\lll}, \ldots$
- $\Diamond$ corresponds to $I^{\pm}$, $J^{\pm}$, ...



First-order vs. Modal formulas

Property of R	Frame condition	Modal axiom	Label	
Reflexive	xRx	$p \to \Diamond p$	**T**	**S4**
Transitive	$xRyRz \Rightarrow xRz$	$\Diamond\Diamond p \to \Diamond p$	**4**	
Serial	$\forall x\, \exists y\; xRy$	$\Box p \to \Diamond p$	**D**	
Dense	$xRy \Rightarrow \exists z\; (xRzRy)$	$\Diamond p \to \Diamond\Diamond p$	ad	
2-dense	$xRy, xRz \Rightarrow \exists u\; (xRu, uRy, uRz)$	$\Diamond p \land \Diamond q \to \Diamond(\Diamond p \land \Diamond q)$	ad_2	
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Important for us is the logic:

$$\mathbf{S4.2} = \mathbf{S4} + (.2)$$

of reflexive, transitive, confluent R.

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Similarly, $\mathbf{OI} = \mathbf{D} + \mathbf{4} + ad_2$ is the logic of serial, transitive, 2-dense 'strict orders'. Important here:

$$\mathbf{OI.2} = \mathbf{OI} + (.2)$$

Minkowski Spacetime + Modal Logic

$$x = (x_0, \vec{x}), \qquad y = (y_0, \vec{y}),$$

where $x_0, y_0 \in \mathbb{R}$, $\vec{x}, \vec{y} \in \mathbb{R}^{n-1}$ and $\|\cdot\|$ is the Euclidean norm.

$$\begin{aligned} x \preccurlyeq y \quad &\text{iff} \quad \|\vec{y} - \vec{x}\| \leq y_0 - x_0, \\ x \ll y \quad &\text{iff} \quad \|\vec{y} - \vec{x}\| < y_0 - x_0. \end{aligned}$$

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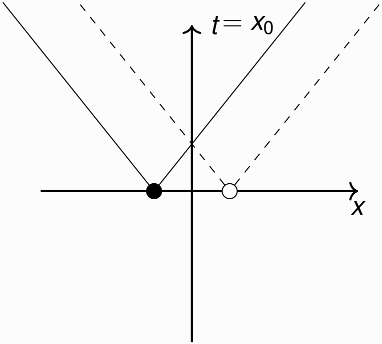

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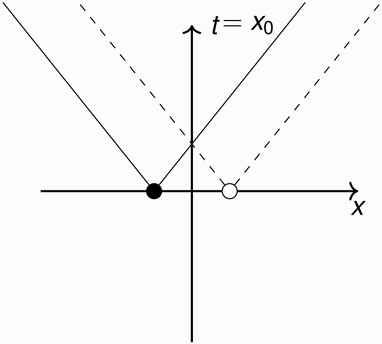

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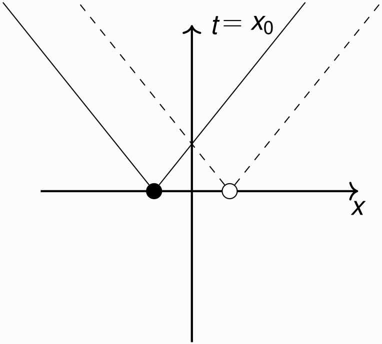

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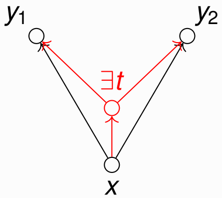
Shapirovsky and Shehtman, "Chronological Future Modality in Minkowski Spacetime", 2002

Lower Bounds from Geometry

- $\ll$ is 2-dense: $\Diamond p \wedge \Diamond q \to \Diamond(\Diamond p \wedge \Diamond q)$;



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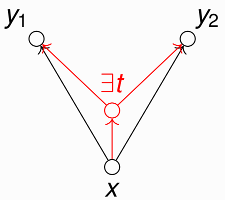

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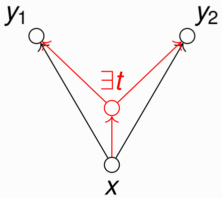

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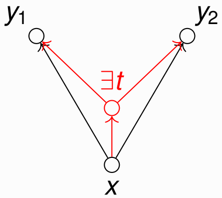

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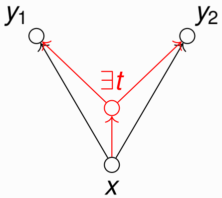

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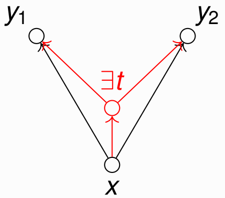

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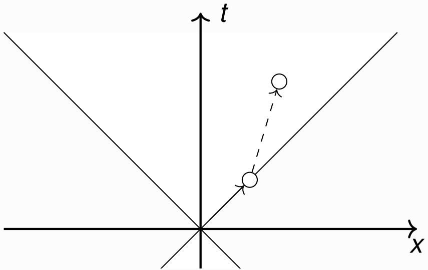

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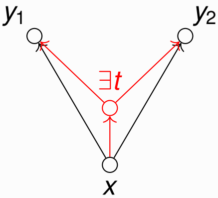

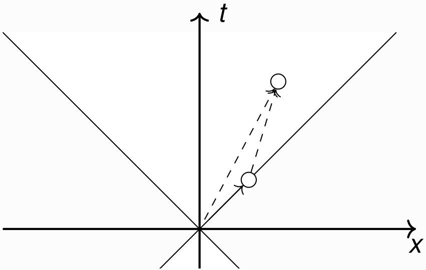

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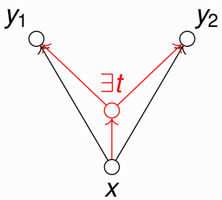

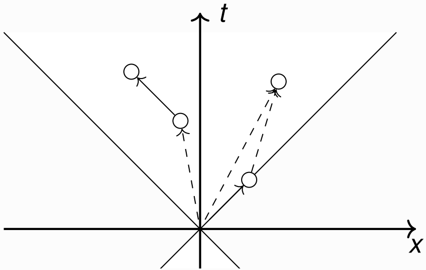

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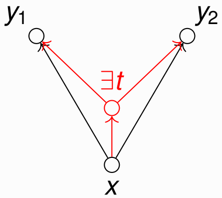

- $\lll$ is 2-dense: $\Diamond p \wedge \Diamond q \to \Diamond(\Diamond p \wedge \Diamond q)$;
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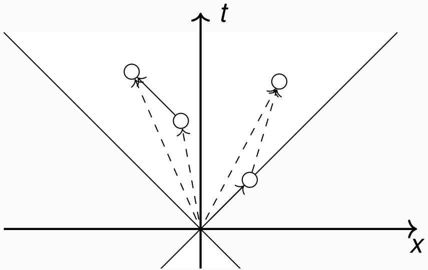


Lower Bounds from Geometry

Corollary (4.1)

Let M be a spacetime. Then:

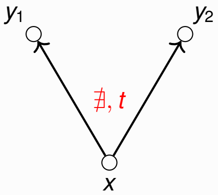
(i) **S4** $\subseteq \mathcal{L}\langle M, \preccurlyeq \rangle$;

(ii) **OI** $\subseteq \mathcal{L}\langle M, \ll \rangle$;

(iii) **D4** $+$ *ad* $\subseteq \mathcal{L}\langle M, \alpha \rangle$.

Why the After Formula?

α is not 2-dense:



For y_1, y_2 on different light rays, the only common causal interpolant from x is x itself.

But α is strict: $x \not\triangleleft x$.

– The 'after formula' is a minimal modification of 2-density that α satisfies;

– Thought to be important in characterising the modal logic of α.

The After Formula

$$a\alpha f := \Diamond(\Diamond(p_1 \wedge \neg p_2 \wedge \Box\neg p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \Box\neg p_1)) \wedge \Diamond q$$
$$\to \Diamond(\Diamond p_1 \wedge \Diamond q) \vee \Diamond(\Diamond p_2 \wedge \Diamond q).$$

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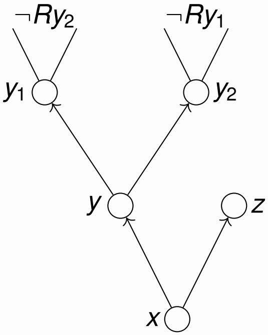

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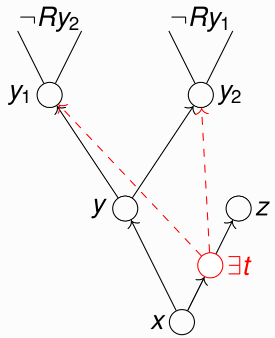
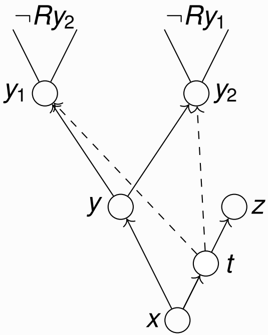


After Formula in Minkowski Spacetime

$$\langle \mathbb{R}^n, \alpha \rangle \models a\alpha f$$

Shapirovsky and Shehtman, “Modal Logics of Regions and Minkowski Spacetime”, 2005



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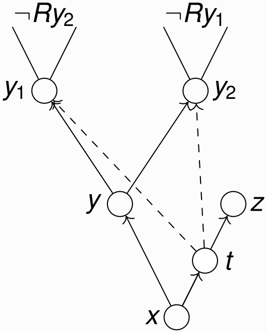

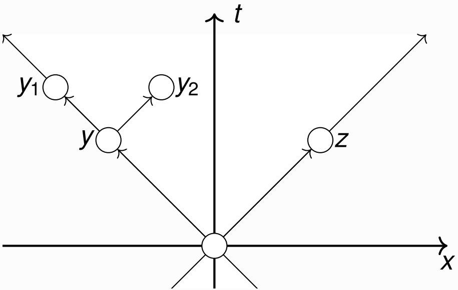

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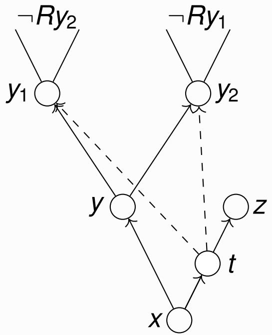

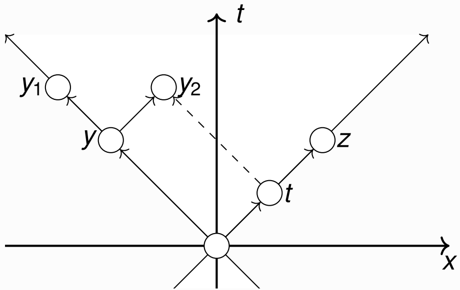

After Formula Globally?

Known:

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The after formula is valid in Minkowski spacetime.

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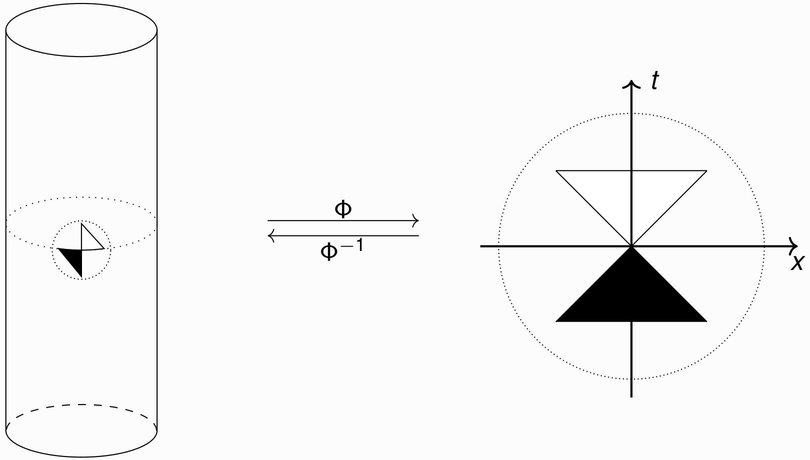
Main question

$$\langle M, \alpha \rangle \models a\alpha f \ ?$$

Does curvature or global topology produce a counterexample?

Geodesic Normal Coordinates

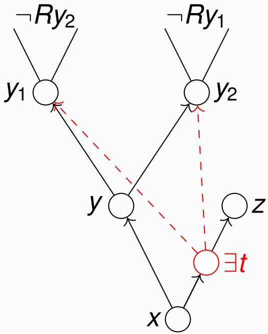


After Formula in General Spacetime

Want to show for a general spacetime, M with α:

$$\langle M, \alpha \rangle \models a\alpha f$$

– Have $x \ \alpha \ y \ \alpha \ y_1, y_2$, $x \ \alpha \ z$; want t such that $x \ \alpha \ t \ \alpha \ y_i, z$



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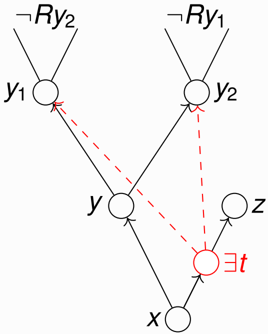

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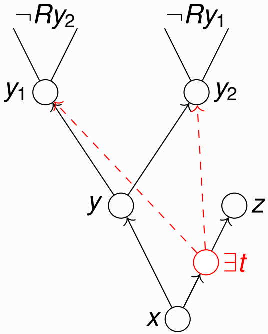

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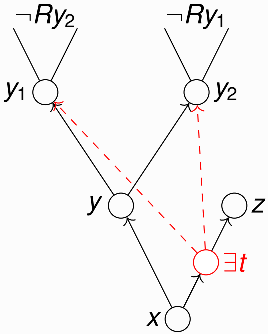
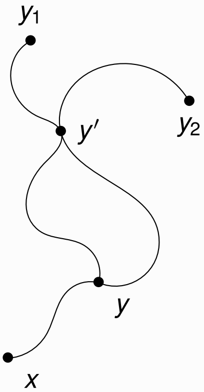
- In case where $x \ll y \ \alpha \ y_1, y_2$ can use Push-up Rule to get $x \ll y_1$
- Main case to consider: $x \to y \to y_1, y_2$, $x \to z$



After Formula in General Spacetime

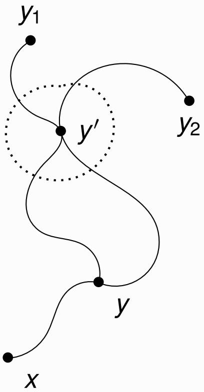
– Consider $x \to y \to y_1, y_2,\ x \to z$



Global Spacetime

After Formula in General Spacetime

- Consider $x \to y \to y_1, y_2,\ x \to z$
- Take neighbourhood around last common point (y')



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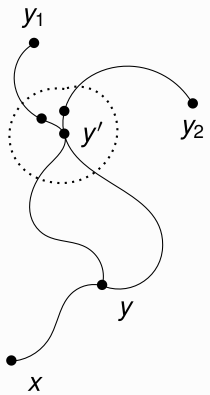

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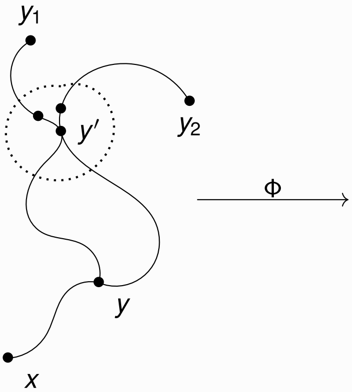

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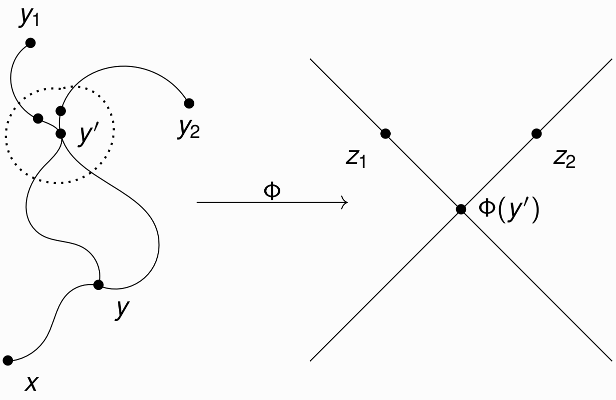

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Local Minkowskian

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- Consider $x \to y \to y_1, y_2,\ x \to z$
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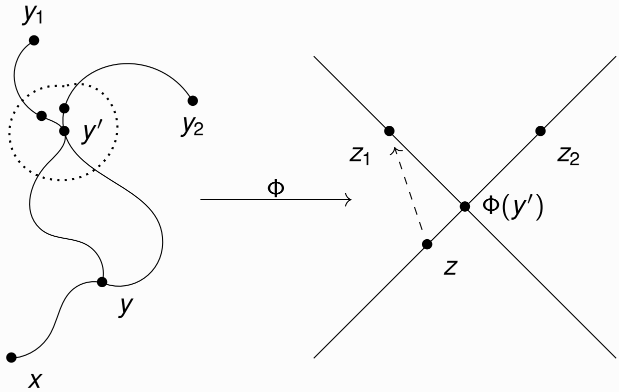

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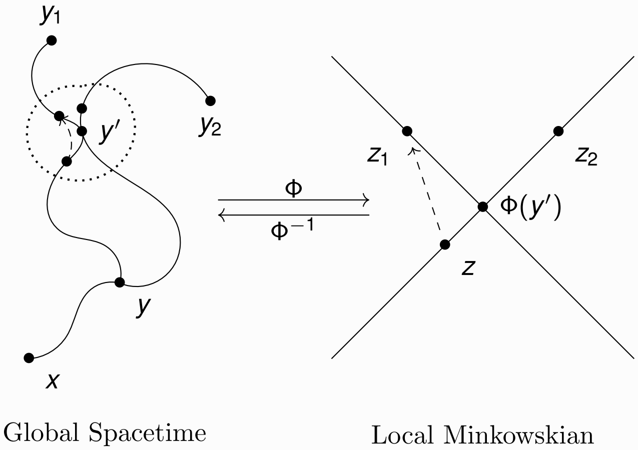

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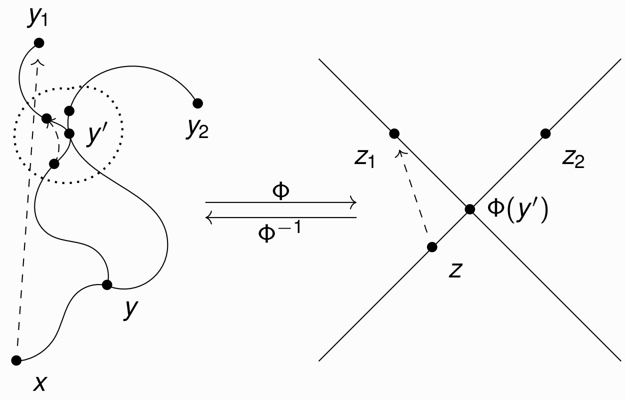

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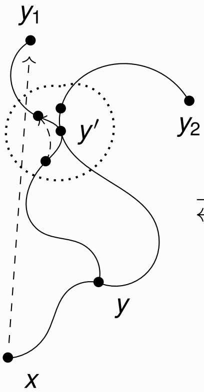
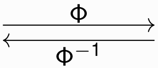
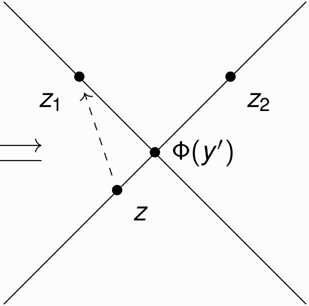
Obtain $\langle M, \alpha \rangle \models a \alpha f$







2-D After Formula

$$
\begin{aligned}
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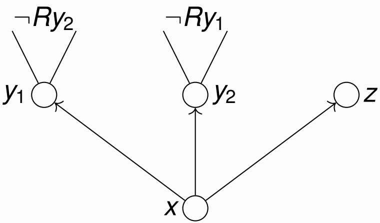

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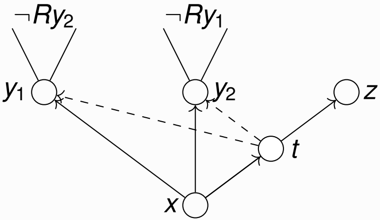

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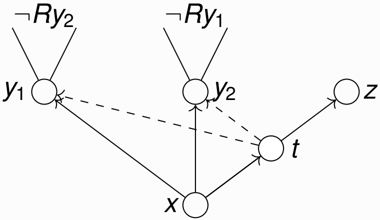

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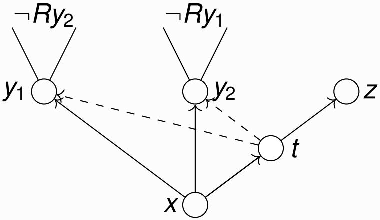

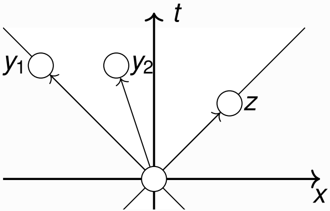

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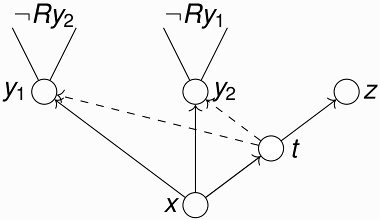

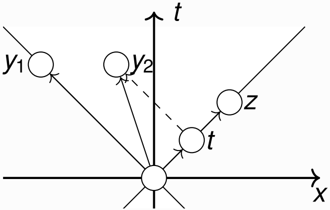

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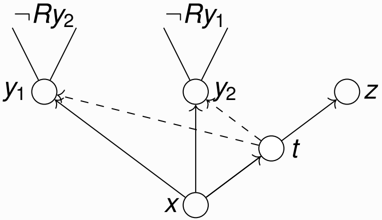

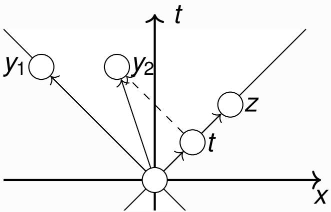

Shapirovsky and Shehtman, "Modal Logics of Regions and Minkowski Spacetime", 2005

2-D After Formula for General Spacetimes

For a general 2-dimensional spacetime M:

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Proof tactic is similar to the after formula, using the fact that there are only two lightlines in 2-dimensional Minkowski space

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- Known: $n \geq 3$: $\langle \mathbb{R}^n, \alpha \rangle \not\models a\alpha_2 f$;
- We conjecture: for arbitrary $n \geq 3$-dimensional spacetime: $\langle M, \alpha \rangle \not\models a\alpha_2 f$

Open Questions

Modal Logic of α in Minkowski space

$$\mathcal{L}\langle \mathbb{R}^2, \alpha \rangle = \ \textbf{D4} + ad + a\alpha_2 f + ?$$

$$\mathcal{L}\langle \mathbb{R}^n, \alpha \rangle = ?$$

Modal link with Push-up Rule

$$x \ll y \preccurlyeq z \implies x \ll z$$

$$x \preccurlyeq y \ll z \implies x \ll z$$

Corresponds to formulae in modal logic?

Upper bound?

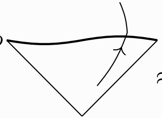
For M non-totally vicious spacetime:

$$\mathcal{L}\langle M, \preccurlyeq \rangle \subseteq \textbf{S4.2}?$$

(similarly for $\ll$ with **OI.2**, α with ?.**2**)

Extensions to more expressive logics (CTL, tense logic, ...)

Domain of ϕ Dependence



$\approx \textbf{AF}\phi$

The Causal Ladder

$$
\begin{tikzcd}
 & \text{Non-totally Vicious} & \\
\text{Causally Non-totally Vicious (cNTV)} \arrow[ur, Rightarrow] & & \text{Chronological} \arrow[ul, Rightarrow] \\
 & \text{cNTV and Chronological} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \\
 & \text{Causal} \arrow[u, Rightarrow] & \\
 & \text{(Past-/Future-) Distinguishing} \arrow[u, Rightarrow] & \\
 & \text{Strongly Causal} \arrow[u, Rightarrow] & \\
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Non-totally Vicious

(α is not reflexive)

$\Rightarrow$ $\Leftarrow$

($\ll$ is irreflexive)

Causally Non-totally Vicious (cNTV)

Chronological

$\Leftarrow$ $\Rightarrow$

cNTV and Chronological

$\Uparrow$

Causal (α irreflexive)

$\Uparrow$

(Past-/Future-) Distinguishing

(past/future of points are distinct)

$\Uparrow$

Strongly Causal

$\Uparrow$

Stably Causal

$\Uparrow$

Globally Hyperbolic

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Non-totally Vicious

(α is not reflexive)
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($\ll$ is irreflexive)
Chronological

cNTV $\Leftarrow$ cNTV and Chronological $\Rightarrow$ Chronological

cNTV and Chronological
$\Uparrow$
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$\Uparrow$
Strongly Causal
$\Uparrow$
Stably Causal (no modal formula)
$\Uparrow$
Globally Hyperbolic

Modal Logics on the Causal Ladder

D4da $:=$ **D4** $+$ *ad* $(\Diamond p \to \Diamond\Diamond p) + a\alpha f$	Spacetime Modal Logic Containment $(L = \mathcal{L}\langle M, \triangleleft\rangle)$		
Spacetime Type (M)	$\ll$	α	$\lll$ / $\preccurlyeq$
No class			
Non-Totally Vicious			
Chronological			
Causally Non-Totally Vicious			
cNTV and Chronological			
Casual			
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

Modal Logics on the Causal Ladder

$\mathbf{D4da} := \mathbf{D4} + ad\ (\Diamond p \to \Diamond\Diamond p) + a\alpha f$	Spacetime Modal Logic Containment $(L = \mathcal{L}\langle M, \lhd \rangle)$		
Spacetime Type (M)	$\ll$	α	$\leqq$ / $\preccurlyeq$
No class	$\mathbf{S4} \subseteq L$	$\mathbf{S4} \subseteq L$	$\mathbf{S4} \subseteq L$
Non-Totally Vicious			
Chronological			
Causally Non-Totally Vicious			
cNTV and Chronological			
Casual			
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

Modal Logics on the Causal Ladder

D4da $:=$ **D4** $+$ ad $(\Diamond p \to \Diamond\Diamond p)$ $+$ $a\alpha f$	Spacetime Modal Logic Containment $(L = \mathcal{L}\langle M, \triangleleft \rangle)$		
Spacetime Type (M)	$\ll$	α	$\underline{\ll}$ / $\preccurlyeq$
No class	**S4** $\subseteq L$	**S4** $\subseteq L$	**S4** $\subseteq L$
Non-Totally Vicious	**OI** $\subseteq L$		
Chronological			
Causally Non-Totally Vicious			
cNTV and Chronological			
Casual			
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

Modal Logics on the Causal Ladder

D4da $:=$ **D4** $+$ ad $(\Diamond p \to \Diamond\Diamond p)$ $+$ $a\alpha f$	Spacetime Modal Logic Containment $(L = \mathcal{L}\langle M, \lhd \rangle)$		
Spacetime Type (M)	$\ll$	α	$\underline{\ll}$ / $\preccurlyeq$
No class	**S4** $\subseteq L$	**S4** $\subseteq L$	**S4** $\subseteq L$
Non-Totally Vicious	**OI** $\subseteq L$		
Chronological	**OI** $+$ (Gabb) $\subseteq L$		
Causally Non-Totally Vicious			
cNTV and Chronological			
Casual			
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

Modal Logics on the Causal Ladder

$\textbf{D4da} := \textbf{D4} + ad\ (\Diamond p \to \Diamond\Diamond p) + a\alpha f$	Spacetime Modal Logic Containment $(L = \mathcal{L}\langle M, \lhd\rangle)$		
Spacetime Type (M)	$\ll$	α	$\leqslant\ /\ \preccurlyeq$
No class	$\textbf{S4} \subseteq L$	$\textbf{S4} \subseteq L$	$\textbf{S4} \subseteq L$
Non-Totally Vicious	$\textbf{OI} \subseteq L$		
Chronological	$\textbf{OI} + (\text{Gabb}) \subseteq L$		
Causally Non-Totally Vicious	$\textbf{OI}$	$\textbf{D4da} \subseteq L$	
cNTV and Chronological			
Casual			
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

Modal Logics on the Causal Ladder

D4da $:=$ **D4** $+ \, ad \, (\Diamond p \to \Diamond\Diamond p) + a\alpha f$	Spacetime Modal Logic Containment ($L = \mathcal{L}\langle M, \triangleleft\rangle$)		
Spacetime Type (M)	$\ll$	α	$\leqslant$ / $\preccurlyeq$
No class	**S4** $\subseteq L$	**S4** $\subseteq L$	**S4** $\subseteq L$
Non-Totally Vicious	**OI** $\subseteq L$		
Chronological	**OI** $+$ (Gabb) $\subseteq L$		
Causally Non-Totally Vicious	**OI**	**D4da** $\subseteq L$	
cNTV and Chronological	**OI** $+$ (Gabb) $\subseteq L$		
Casual			
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

Modal Logics on the Causal Ladder

D4da $:=$ **D4** $+$ *ad* $(\Diamond p \to \Diamond\Diamond p) + a\alpha f$	Spacetime Modal Logic Containment $(L = \mathcal{L}\langle M, \lhd\rangle)$		
Spacetime Type (M)	$\ll$	α	$\underline{\ll}$ / $\preccurlyeq$
No class	**S4** $\subseteq L$	**S4** $\subseteq L$	**S4** $\subseteq L$
Non-Totally Vicious	**OI** $\subseteq L$		
Chronological	**OI** $+$ (Gabb) $\subseteq L$		
Causally Non-Totally Vicious	**OI**	**D4da** $\subseteq L$	
cNTV and Chronological	**OI** $+$ (Gabb) $\subseteq L$		
Casual		**D4da** $+$ (Gabb) $\subseteq L$	
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic			

Modal Logics on the Causal Ladder

D4da $:=$ **D4** $+$ *ad* $(\Diamond p \to \Diamond\Diamond p) + a\alpha f$	Spacetime Modal Logic Containment $(L = \mathcal{L}\langle M, \lhd \rangle)$		
Spacetime Type (M)	$\ll$	α	$\leqq$ / $\preccurlyeq$
No class	**S4** $\subseteq L$	**S4** $\subseteq L$	**S4** $\subseteq L$
Non-Totally Vicious	**OI** $\subseteq L$		
Chronological	**OI** $+$ (Gabb) $\subseteq L$		
Causally Non-Totally Vicious	**OI**	**D4da** $\subseteq L$	
cNTV and Chronological	**OI** $+$ (Gabb) $\subseteq L$		
Casual		**D4da** $+$ (Gabb) $\subseteq L$	
Distinguishing			
$\vdots$	$\vdots$		
Globally hyperbolic	**OI** $+$ (Gabb) $\subseteq L$	**D4da** $+$ (Gabb) $\subseteq L$	**S4** $\subseteq L$