Localic Esakia Duality

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`https://nvds.site/`

Preprint: coming soon!

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2026 Logic Colloquium

Recap: Stone dualities

X	Stone spaces	$\leftsquigarrow\!\!\rightsquigarrow$	Boolean algebras

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$(X, \leqslant)$	Priestley spaces	$\leftrightsquigarrow$	Distributive lattices
$\Downarrow$	*impose property: $\downarrow$ preserves (cl)opens*		
$(X, \leqslant)$	Esakia spaces	$\leftrightsquigarrow$	Heyting algebras

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$\Downarrow$	*impose property: $\downarrow$ preserves (cl)opens*		
$(X, \leqslant)$	Esakia spaces	$\leftrightsquigarrow$	Heyting algebras

problem: **not constructive** $\qquad A \cong \mathrm{ClopUp}(\mathrm{Spec}(A))$

Constructive?

[Tow97] C. Townsend, *“Localic Priestley Duality”*, 1997.

Constructive?

$(X, \leqslant)$ Priestley ~~spaces~~ *locales* $\overset{\text{[Tow97]}}{\leftrightsquigarrow}$ ~~Distributive lattices~~ *coherent frames*

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Constructive?

$(X, \leqslant)$ Priestley ~~spaces~~ *locales* $\overset{\text{[Tow97]}}{\leftsquigarrow\!\rightsquigarrow}$ ~~Distributive lattices~~ *coherent frames*

$(X, \leqslant)$ Esakia ~~spaces~~ *locales* $\underset{?}{\leftsquigarrow\!\rightsquigarrow}$ Heyting ~~algebras~~ *frames*

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Plan

$$
\begin{tikzcd}
\text{Heyting frames} \arrow[rr, leftrightarrow, dashed, "?"] \arrow[dr, leftrightarrow] & & \text{Esakia locales} \arrow[dl, leftrightsquigarrow] \\
& \text{Esakia frames} &
\end{tikzcd}
$$

Frames

frame/locale complete lattice L where $\wedge$ distributes over $\bigvee$

[Joh82] P. T. Johnstone, *Stone Spaces*, 1982.
[BCM23] G. Bezhanishvili et al., *“A Frame-Theoretic Perspective on Esakia Duality”*, 2023.

Frames

frame/locale complete lattice L where $\wedge$ distributes over $\bigvee$

algebraic join-basis $K(L)$

compact elements

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Heyting coherent + $K(L)$ Heyting [BCM23]

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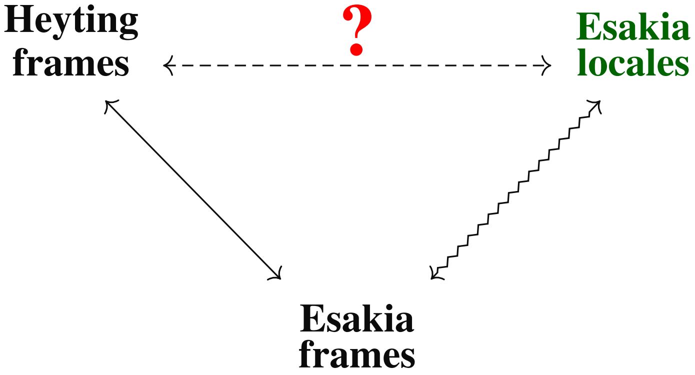
Esakia locales



Priestley locales [Tow97]

Priestley locale

Stone locale X + closed <u>localic</u> partial order $\leqslant$ s.t.

$$u_{\leqslant} = \bigvee\{c \otimes \neg c : c \in K(\mathcal{O}X) : {\downarrow}\neg c = \neg c\}.$$

$x \nleqslant y \implies \exists$separating clopen upset

Esakia locales

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Esakia locale

Priestley locale with

open source projection $s\colon {\leqslant} \to X$

${\downarrow}$preserves (cl)opens

Esakia frames

$$\begin{tikzcd}
\textbf{Heyting frames} \arrow[rr, dashed, leftrightarrow, "\textbf{?}"] \arrow[dr, leftrightarrow] & & \textbf{Esakia locales} \arrow[dl, leftrightsquigarrow] \\
& \textbf{Esakia frames} &
\end{tikzcd}$$

Conic frames

localic $\leqslant$ is hard to work with…

[vdS26] N. van der Schaaf, *Localic Relations with Open Cones*, 2026. `arXiv:2605.04038`.

Conic frames

localic $\leqslant$ is hard to work with…

Theorem. **(cf. [vdS26])**

$$\begin{tikzcd} \mathbf{rosLoc} \arrow[r, shift left, "\mathsf{cone}"] \arrow[r, phantom, "\perp"] & \nabla\mathbf{Frm}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{rel}"] \end{tikzcd}$$

arbitrary localic relations with open source

frames L with $\nabla\colon L \to L$

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Esakia frames

(E0) $(E, \triangledown)$ is a conic frame — $\leqslant$ open source

(E1) E is a Stone frame — X Stone locale

(E2) $\triangledown$ is a closure operator — $\leqslant$ preorder

(E3) $(E, \triangledown)$ is closed — $\leqslant$ closed

(E4) $K(E)$ is generated as a Boolean algebra by $K_{\triangledown}(E)$ — $\leqslant$ anti-symmetric (*partial* order)

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$\{c \in K(E) : \triangledown \neg c = \neg c\}$

Esakia (Locales $\leftrightsquigarrow$ Frames)

Theorem. Restricting $\mathsf{cone} \dashv \mathsf{rel}$ gives:

$$\text{Esakia locales} \quad \leftrightsquigarrow \quad \text{Esakia frames}$$

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Esakia Frames $\to$ Heyting Frames

$$\mathrm{Idl}(A) \text{ is Heyting} \quad \overset{\text{[BCM23]}}{\iff} \quad A \text{ is Heyting}$$

[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Esakia Frames $\to$ Heyting Frames

$$\mathrm{Idl}(A) \text{ is Heyting} \quad \overset{\text{[BCM23]}}{\Longleftrightarrow} \quad A \text{ is Heyting}$$

$$\begin{aligned} (E, \nabla) \quad &\longmapsto \quad K_\nabla(E) \\ & a \to b := \neg \nabla(a \wedge \neg b) \end{aligned}$$

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$$U \to V := X \setminus {\Downarrow}(U \setminus V)$$

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$$a \to b := \neg \nabla(a \wedge \neg b)$$

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So $\mathrm{Idl}(K_\nabla(E))$ is a Heyting frame

[BCM23] G. Bezhanishvili et al., *"A Frame-Theoretic Perspective on Esakia Duality"*, 2023.

Heyting Frames $\to$ Esakia Frames

H

Heyting Frames $\to$ Esakia Frames

$$H \quad \longmapsto \quad K(H)$$

Heyting Frames $\to$ Esakia Frames

$$H \quad \longmapsto \quad K(H) \quad \longmapsto \quad \mathsf{Bool}(K(H))$$

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free Boolean algebra
on distributive lattice

Heyting Frames $\to$ Esakia Frames

constructively [Per57]

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[Per57] W. Peremans, *“Embedding of a Distributive Lattice into a Boolean Algebra”*, 1957.

Heyting Frames $\to$ Esakia Frames

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[Per57]

$$H \quad \longmapsto \quad K(H) \quad \longmapsto \quad \mathsf{Bool}(K(H)) \quad \longmapsto \quad \mathrm{Idl}(\mathsf{Bool}(K(H)))$$

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Heyting frames $\to$ Esakia frames

$\mathrm{Idl}(\mathsf{Bool}(K(H)))$ is generated by $u_a \wedge \ell_b \quad a, b \in K(H)$

Heyting frames → Esakia frames

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$A \setminus B$
clopen upsets

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Heyting frames $\to$ Esakia frames

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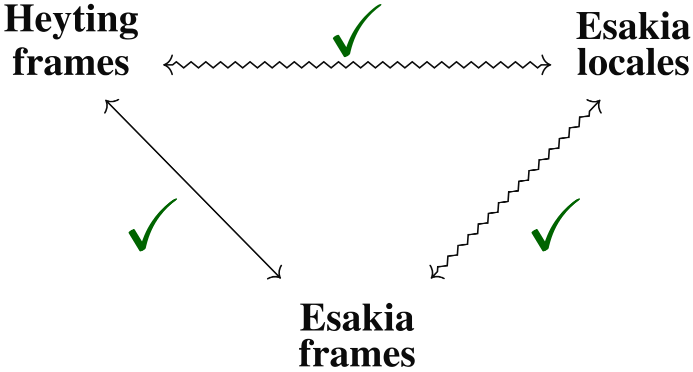
$(\mathrm{Idl}(\mathsf{Bool}(K(H))), \nabla_H)$ is an Esakia frame

(Heyting $\leftrightarrow$ Esakia) frames

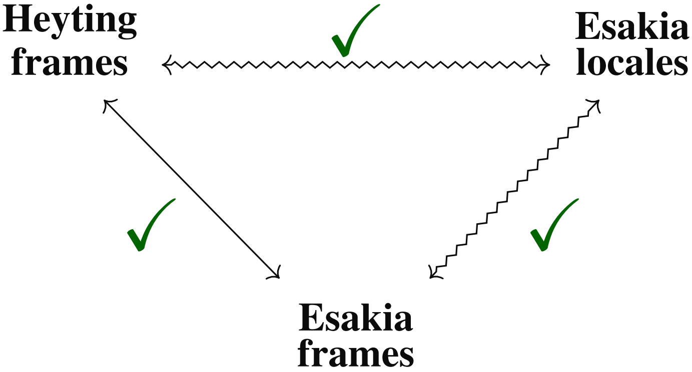
Theorem.

$$\text{Heyting frames} \quad \longleftrightarrow \quad \text{Esakia frames}$$

Conclusion



Conclusion



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Localic relations: **arXiv:2605.04038**

Thanks!

Conclusion

$$
\begin{tikzcd}[column sep=small]
\mathbf{CohFrm}^{\mathrm{op}} \arrow[rrrr, shift left=2] \arrow[rrrr, phantom, "\simeq"] & & & & \mathbf{PLoc} \arrow[llll, shift left=2] \arrow[rr, phantom, "\subseteq"] & & \mathbf{rLoc} \\
\mathbf{HFrm}^{\mathrm{op}} \arrow[u, phantom, sloped, "\subseteq"] \arrow[rrrr, shift left=2] \arrow[rrrr, phantom, "\simeq"] \arrow[dr, shift right=2, swap, "H \longmapsto (\mathrm{Idl}(B_H),\, \triangledown_H)"] \arrow[dr, phantom, sloped, "\simeq"] & & & & \mathbf{ELoc} \arrow[u, phantom, sloped, "\subseteq"] \arrow[llll, shift left=2] \arrow[rr, phantom, "\subseteq"] \arrow[dlll, shift left=2, "(\mathcal{O}X,\, \downarrow) \longleftarrow\!\shortmid (X, R)"] \arrow[dlll, phantom, sloped, "\simeq"] & & \mathbf{rosLoc} \arrow[u, phantom, sloped, "\subseteq"] \arrow[dlll, shift right=2, swap, "\mathsf{cone}"] \arrow[dlll, phantom, sloped, "\perp"] \\
& \mathbf{EFrm}^{\mathrm{op}} \arrow[ul, shift right=2, swap, "\mathrm{Idl}(K_{\triangledown}(E)) \longleftarrow\!\shortmid (E,\, \triangledown)"] \arrow[urrr, shift left=2, "(E,\, \triangledown) \longmapsto (X,\, R_{\triangledown})"] \arrow[r, phantom, "\subseteq"] & & \triangledown\mathbf{Frm}^{\mathrm{op}} \arrow[urrr, shift right=2, swap, "\mathsf{rel}"] & & &
\end{tikzcd}
$$