Ordered Locales

Birmingham Theory Seminar

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joint work with Chris Heunen

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{arXiv:2303.03813}

to appear in JPAA

The University of Edinburgh

School of Informatics

Overview

Locales

Ordered locales

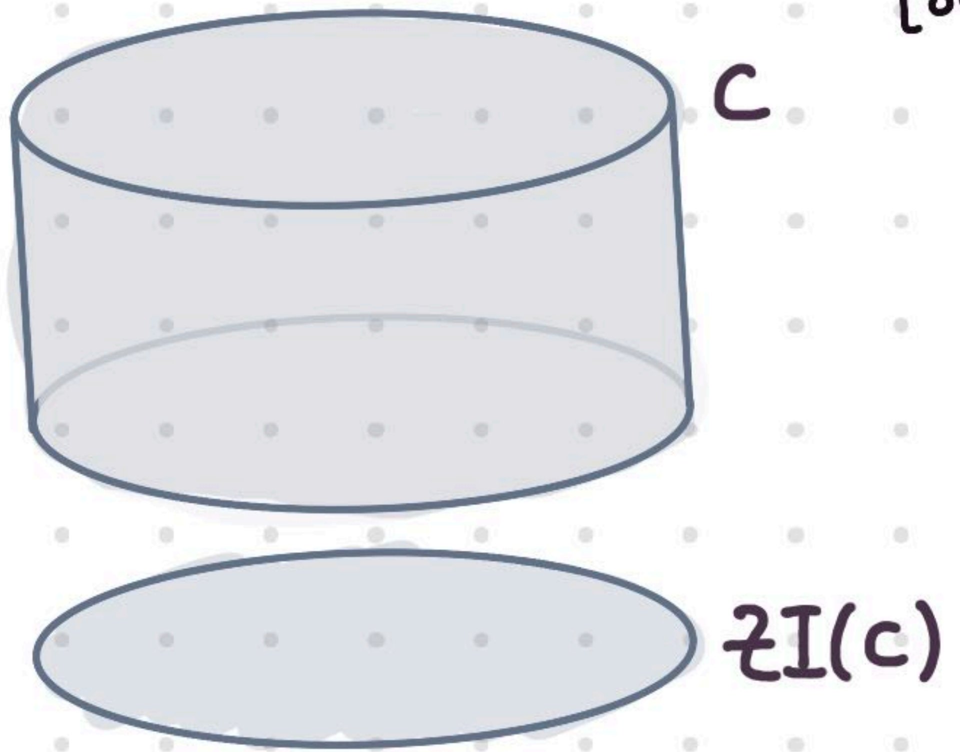
Adjunction

Future work

Motivation

Spacetimes e.g. [Forrest 96]

tensor topology [Moliner, Heunen, Tull 20]
[Soares Barbosa, Heunen 23]



EXAMPLES

$$ZI(\mathrm{Sh}(S)) \cong \mathcal{O}S$$

$$ZI(\mathrm{Hilb}_{C_0(S)}) \cong \mathcal{O}S$$

Idea

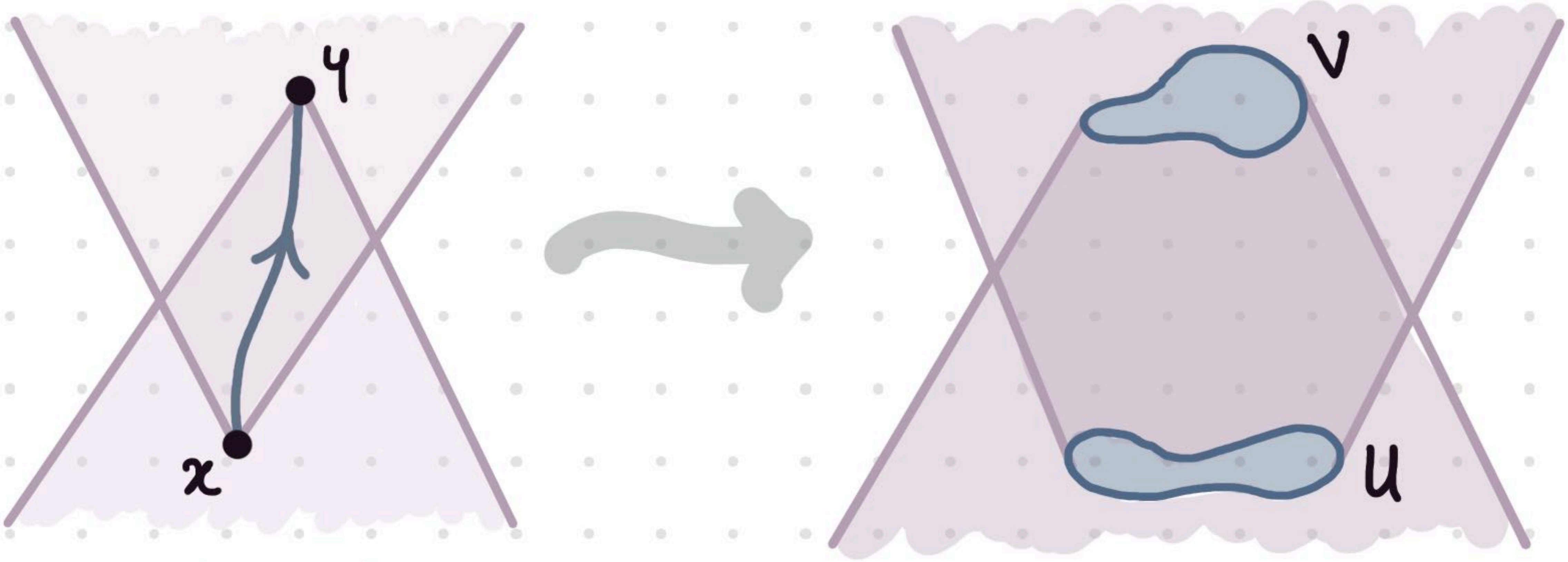
$$\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[u, hook] \arrow[l, shift left, "\mathrm{pt}"]
\end{tikzcd}$$

Causality

+

spaces

Idea



causal order

$x \leq y$

region causality

$U \trianglelefteq V$

$$
\begin{tikzcd}
\mathbf{OrdTop} \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left=1.5] \\
\mathbf{Top} \arrow[u, hook] \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[u, hook] \arrow[l, shift left=1.5]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[u, hook] \arrow[l, shift left]
\end{tikzcd}
$$

Locales

$$\mathbf{Top} \longrightarrow \mathbf{Frm}^{op}$$
$$S \longmapsto \mathcal{O}S$$
$$(S \xrightarrow{f} T) \longmapsto (\mathcal{O}T \xrightarrow{f^{-1}} \mathcal{O}S)$$

Frames as algebraic dual to a type of space:

$$\mathbf{Loc} := (\mathbf{Frm})^{op}$$

LOCALES

object: $X \in \mathbf{Loc} \longleftrightarrow \mathcal{O}X \in \mathbf{Frm}$

arrows: $(X \xrightarrow{f} Y) \in \mathbf{Loc} \longleftrightarrow (\mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X) \in \mathbf{Frm}$

Locales

Not every locale comes from a topological space!! Yet:

POINT: map of locales:

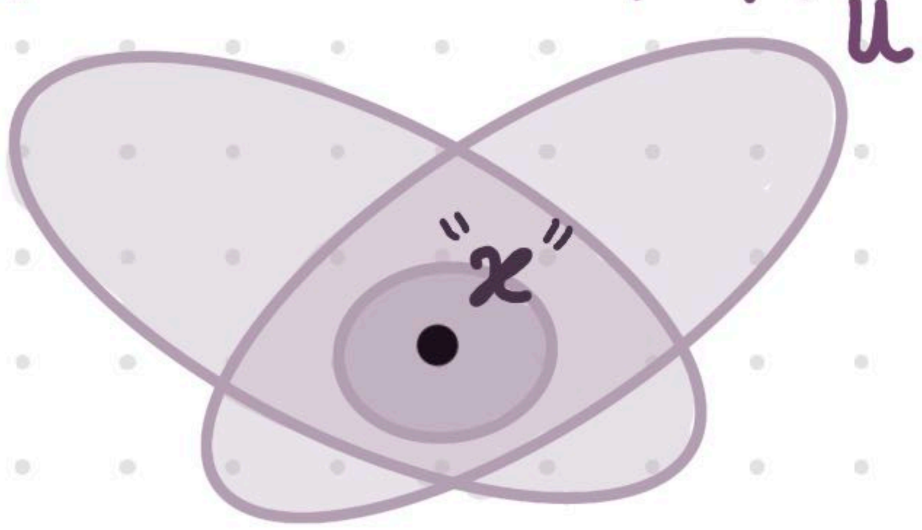
$$p : 1 \longrightarrow X$$

$$\mathcal{O}X \xrightarrow{p^{-1}} \mathcal{O}1 := \{F < T\}$$

$$\Updownarrow$$

$$\mathcal{F} = \{U \in \mathcal{O}X : p^{-1}(U) = T\}$$

$$\begin{array}{ccc} \mathbf{Loc} & \xrightarrow{\;pt\;} & \mathbf{Top} \\ X & \longmapsto & pt(X) \\ f & \longmapsto & pt(f) \end{array}$$



$$U \in \mathcal{F} \iff \text{"}x\text{"} \in U$$

Locales

THEOREM

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left=2, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, shift left=2, "\mathrm{pt}"] \end{tikzcd}$$

Locales

Start with $S \in \mathbf{Top}$.

REG. O.

$$\mathcal{R}S := \{ U \in \mathcal{O}S : U = (\overline{U})^\circ \}$$

defines $\mathrm{Reg}(S) \in \mathbf{Loc}$.

If S is Hausdorff:

LEMMA

$$\mathrm{Pt}\,\mathrm{Reg}(S) \cong \{ \text{isolated points in } S \}$$

E.G. $\mathrm{Pt}\,\mathrm{Reg}(\mathbb{R}^n) = \emptyset$

$$
\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[l, shift left] \arrow[u, hook]
\end{tikzcd}
$$

Ordered Spaces

ORD. SP. A space S with a preorder $\leq$.

MAPS continuous monotone functions $f : S \longrightarrow T$.

$x \leq y \implies f(x) \leq f(y)$.

We get a category:

OrdTop

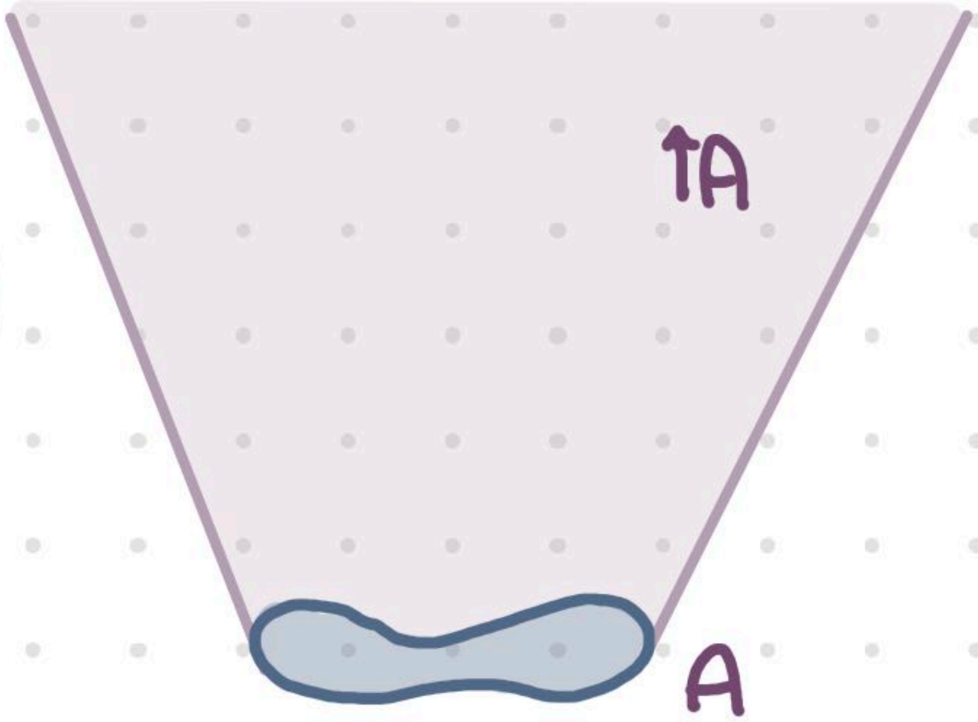
Ordered Spaces

capture $\leq$ in terms of open regions?

CONES

the future cone of $A \subseteq S$:

$$\uparrow A := \{ x \in S \mid \exists a \in A : a \leq x \}.$$



LEMMA

- $A \subseteq B \implies \uparrow A \subseteq \uparrow B$
- $A \subseteq \uparrow A$
- $\uparrow\uparrow A \subseteq \uparrow A$

$$x \leq y \iff y \in \uparrow\{x\}$$

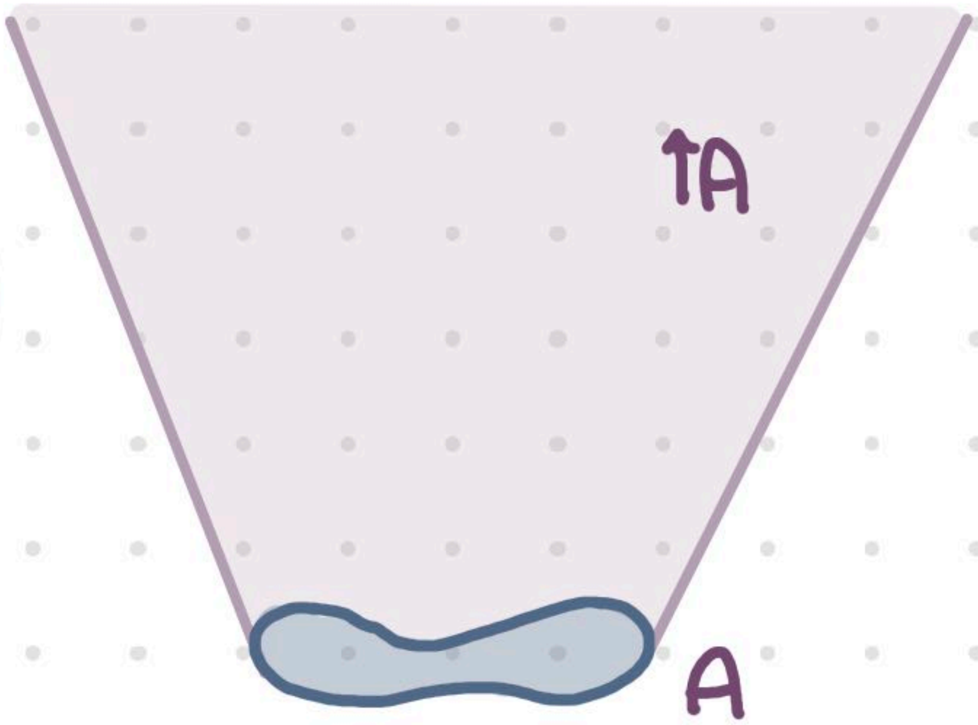
Ordered Spaces

capture $\leqslant$ in terms of open regions?

CONES

the future cone of $A \subseteq S$:

$$\uparrow A := \{ x \in S \mid \exists a \in A : a \leqslant x \}.$$



LEMMA

$f : S \longrightarrow T$ monotone iff:

$$\uparrow f^{-1}(B) \subseteq f^{-1}(\uparrow B).$$

Ordered Spaces

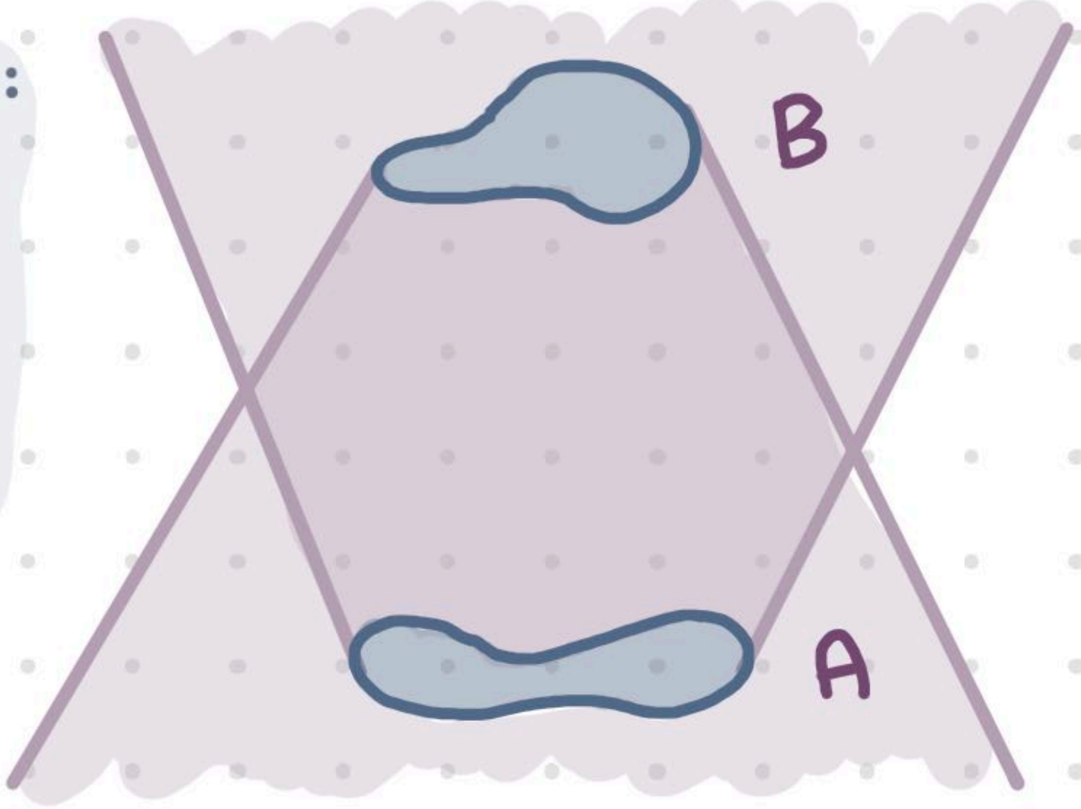
capture $\leqslant$ in terms of open regions?

EM ORDER

If $(S, \leqslant)$ is a preorder:

$$A \trianglelefteq B \iff \begin{cases} A \subseteq \downarrow B, \\ B \subseteq \uparrow A \end{cases}$$

is a preorder on $\mathcal{P}(S)$.



$(\mathrm{Loc}(S), \trianglelefteq)$ as prototypical ordered locale

$$
\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[l, shift left] \arrow[u, hook]
\end{tikzcd}
$$

Ordered Locales

ORD. LOC.

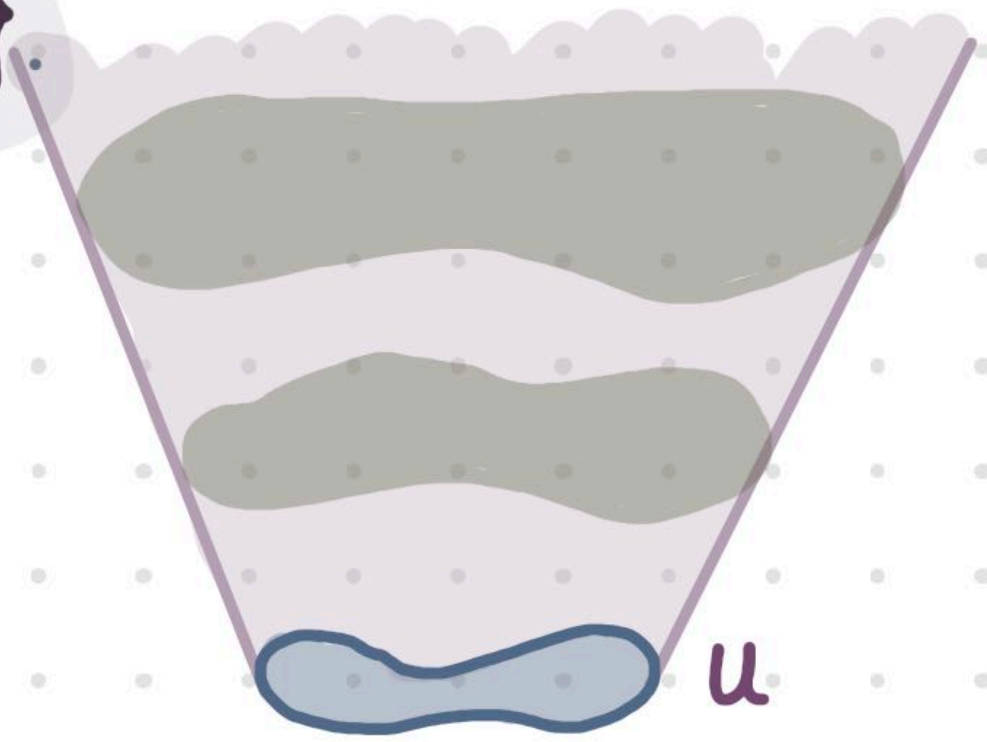
a locale X with preorder $\trianglelefteq$ on $\mathcal{O}X$:

$$\forall i : U_i \trianglelefteq V_i \implies \bigvee U_i \trianglelefteq \bigvee V_i .$$

Ordered Locales

CONES

$$\uparrow U := \bigvee \{ W \in \mathcal{O}X : U \trianglelefteq W \}$$



LEMMA

- $\downarrow U \trianglelefteq U \trianglelefteq \uparrow U$
- $U \subseteq V \implies \uparrow U \subseteq \uparrow V$
- $U \subseteq \uparrow U$
- $\uparrow\uparrow U \subseteq \uparrow U$

$$U \trianglelefteq W,\ V \trianglelefteq V \implies V = U \vee V \trianglelefteq V \vee W$$
$$\implies W \subseteq V \vee W \subseteq \uparrow V.$$

Ordered Locales

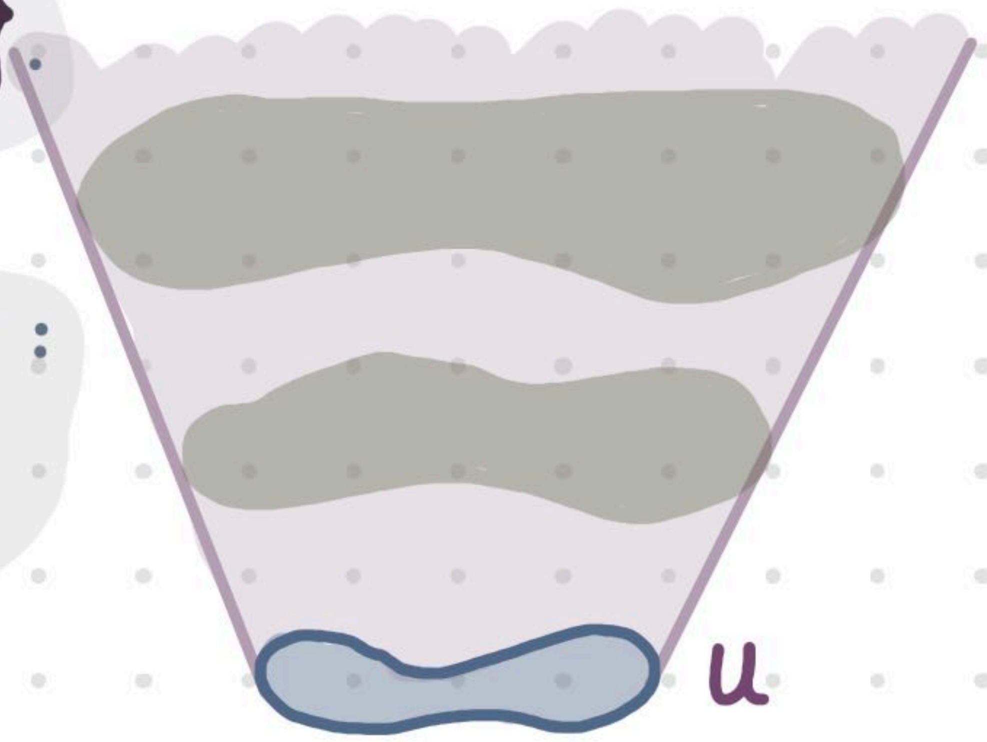
CONES

$$\uparrow U := \bigvee \{ W \in \mathcal{O}X : U \trianglelefteq W \}.$$

MAPS

monotone map $f : X \longrightarrow Y$:

$$\uparrow f^{-1}(U) \subseteq f^{-1}(\uparrow U).$$



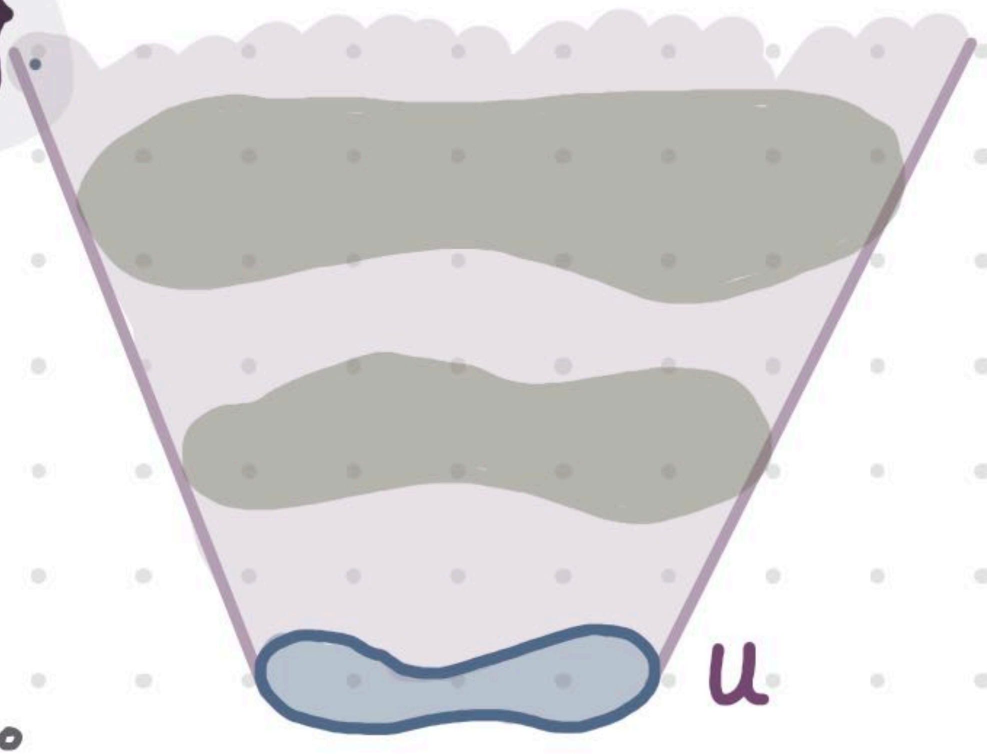
We get a category :

OrdLoc

Ordered Locales

CONES

$$\Uparrow U := \bigvee\{W \in \mathcal{O}X : U \trianglelefteq W\}.$$



E.G.

in a space $(S, \leq)$:

$\Uparrow U = (\uparrow U)^\circ$, $\Downarrow U = (\downarrow U)^\circ$

$x \in \Uparrow U \Rightarrow \exists W \ni x : U \trianglelefteq W \Rightarrow x \in W \subseteq \uparrow U.$

$(\uparrow U)^\circ \subseteq \uparrow U$, $U \subseteq (\uparrow U)^\circ \subseteq \downarrow (\uparrow U)^\circ \Longrightarrow U \trianglelefteq (\uparrow U)^\circ.$

Adjunction

$$\begin{tikzcd} \mathbf{OrdTop} \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left=2] \end{tikzcd}$$

$$(S, \leqslant) \longmapsto (\mathrm{Loc}(S), ?)$$

$$(\mathrm{pt}(X), ?) \longmapsfrom (X, \trianglelefteq)$$

Adjunction

$$
\begin{tikzcd}
\mathbf{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left] \\
(S, \leqslant) \arrow[r, mapsto] & (\mathrm{Loc}(S), \trianglelefteq) \\
(\mathrm{pt}(X), ?) & (X, \trianglelefteq) \arrow[l, mapsto]
\end{tikzcd}
$$

Egli-Milner

Adjunction

OrdTop $\longrightarrow$ **OrdLoc**

$(S, \leqslant) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$

OPEN CONES

$(S, \leqslant)$ has OC if:
$\forall U \in \mathcal{O}S : \uparrow U, \downarrow U \in \mathcal{O}S$

E.G.

- smooth spacetimes
- interval topology of d.latt.
- (co)discrete spaces

LEMMA

S has OC iff:
$T \xrightarrow{g} S$ monotone $\Longrightarrow$
$\mathrm{Loc}(T) \xrightarrow{\mathrm{Loc}(g)} \mathrm{Loc}(S)$ monotone

Adjunction

$$\begin{tikzcd}
\mathbf{OrdTop}_{oc} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left=1.5ex] \\
(S, \leq) \arrow[r, mapsto] & (\mathrm{Loc}(S), \trianglelefteq) \\
(\mathrm{pt}(X), ?) & (X, \trianglelefteq) \arrow[l, mapsto]
\end{tikzcd}$$

Adjunction

$$\mathbf{OrdLoc} \longrightarrow \mathbf{OrdTop}$$
$$(X, \trianglelefteq) \longmapsto (pt(X), \leqslant)$$

pt. ORD.

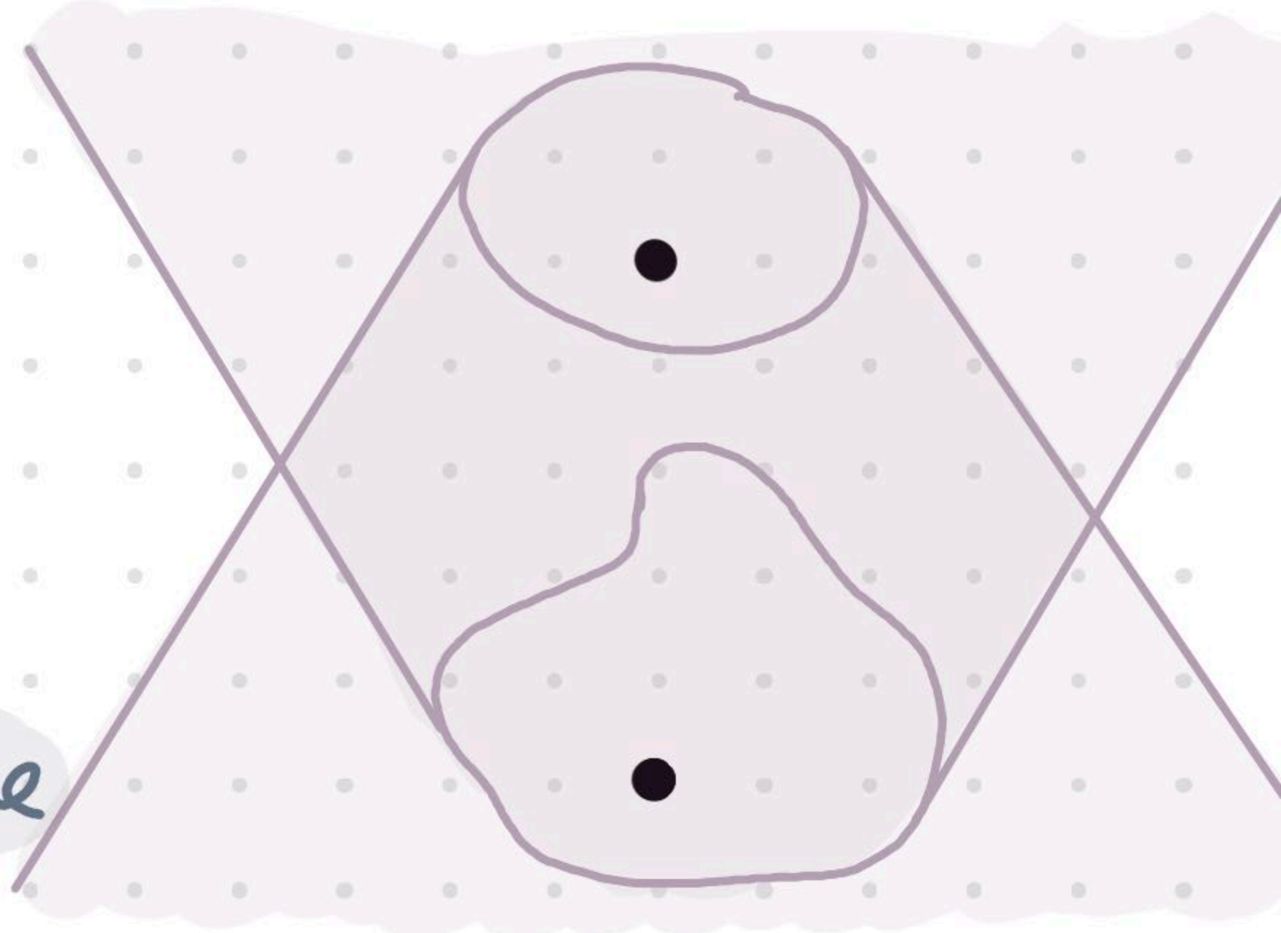
for $\mathcal{F}, \mathcal{G} \in pt(X)$:

$$\mathcal{F} \leqslant \mathcal{G} \iff \begin{array}{l} \forall U \in \mathcal{F} : \uparrow U \in \mathcal{G}, \\ \forall V \in \mathcal{G} : \downarrow V \in \mathcal{F}. \end{array}$$

$$x \leqslant y \iff \begin{array}{lcl} \forall U \ni x & : & y \in \uparrow U, \\ \forall V \ni y & : & x \in \downarrow V. \end{array}$$

LEMMA

f monotone $\Rightarrow$ $pt(f)$ monotone



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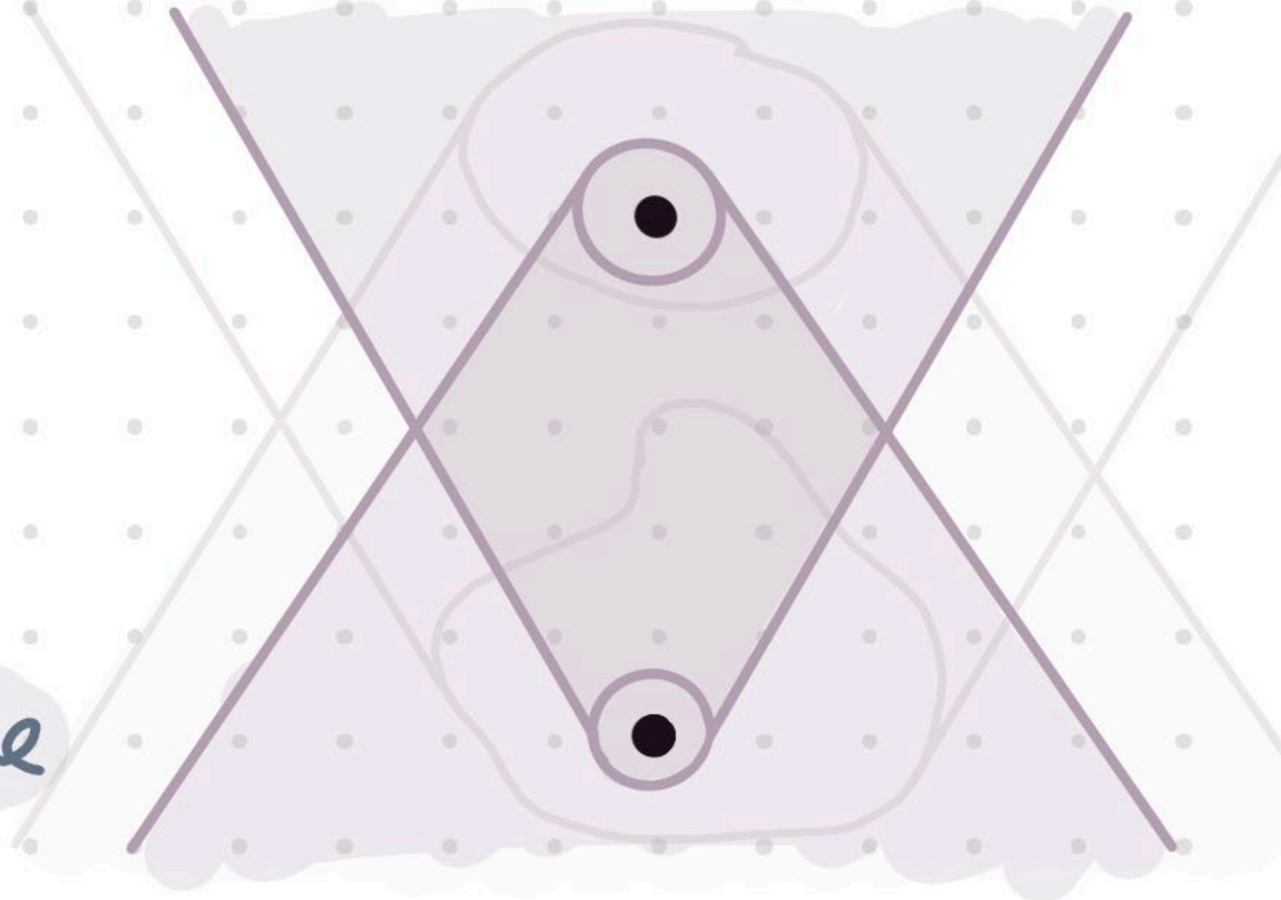

Adjunction

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$$(X, \trianglelefteq) \longmapsto (\mathrm{pt}(X), \leqslant)$$

pt. ORD.

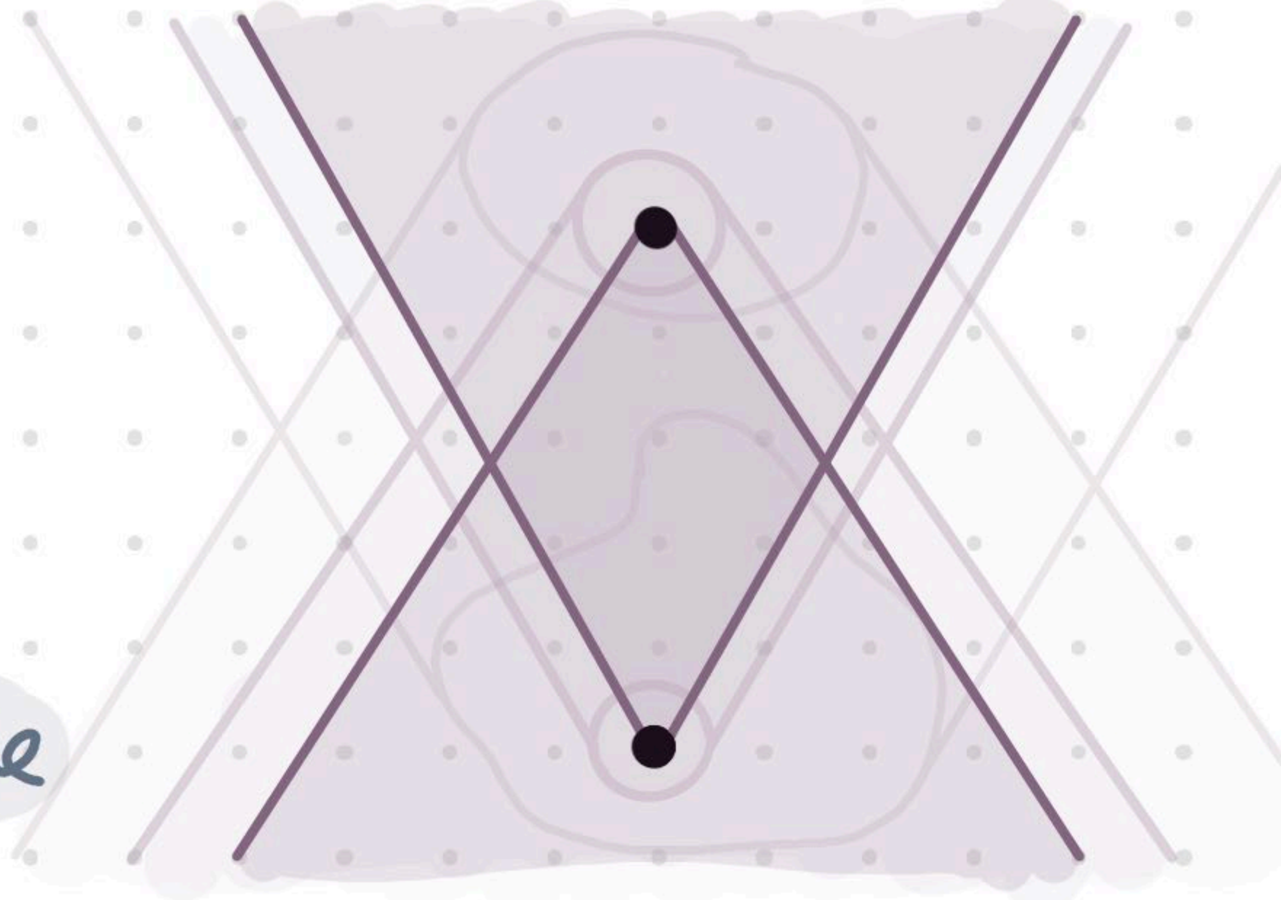
for $\mathcal{F}, \mathcal{G} \in \mathrm{pt}(X)$:

$$\mathcal{F} \leqslant \mathcal{G} \iff \begin{array}{l} \forall U \in \mathcal{F} : {\Uparrow}U \in \mathcal{G}, \\ \forall V \in \mathcal{G} : {\Downarrow}V \in \mathcal{F}. \end{array}$$

$$x \leqslant y \iff \begin{array}{lll} \forall U \ni x & : & y \in {\uparrow}U, \\ \forall V \ni y & : & x \in {\downarrow}V. \end{array}$$

LEMMA

f monotone $\Longrightarrow$ $\mathrm{pt}(f)$ monotone



Adjunction

$$\begin{tikzcd} \mathbf{OrdTop}^{\bullet}_{oc} \arrow[r, shift left=2, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc}^{\bullet} \arrow[l, shift left=2, "\mathrm{pt}"] \end{tikzcd}$$

$$(S, \leq) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$$

$$(\mathrm{pt}(X), \leq) \longmapsfrom (X, \trianglelefteq)$$

THEOREM

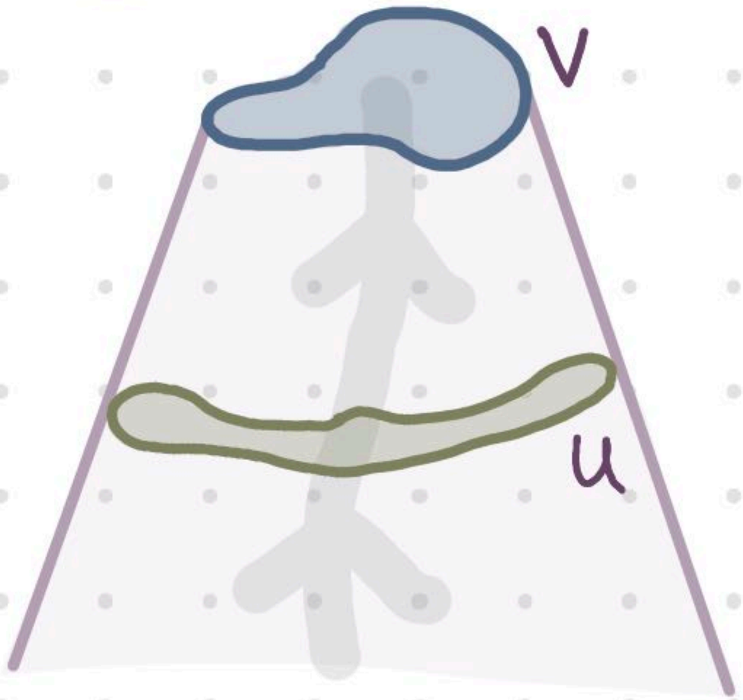
$$\mathbf{OrdTop}_{OC}^{\bullet} \underset{pt}{\overset{Loc}{\underset{\perp}{\rightleftarrows}}} \mathbf{OrdLoc}^{\bullet}$$

COROLLARY

$$\left\{\begin{array}{c}\text{sober } T_0\text{-ordered} \\ \text{spaces with OC}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{spatial ordered} \\ \text{locales with } \bullet\end{array}\right\}$$

Causal Coverage

[Christensen, Crane 05]



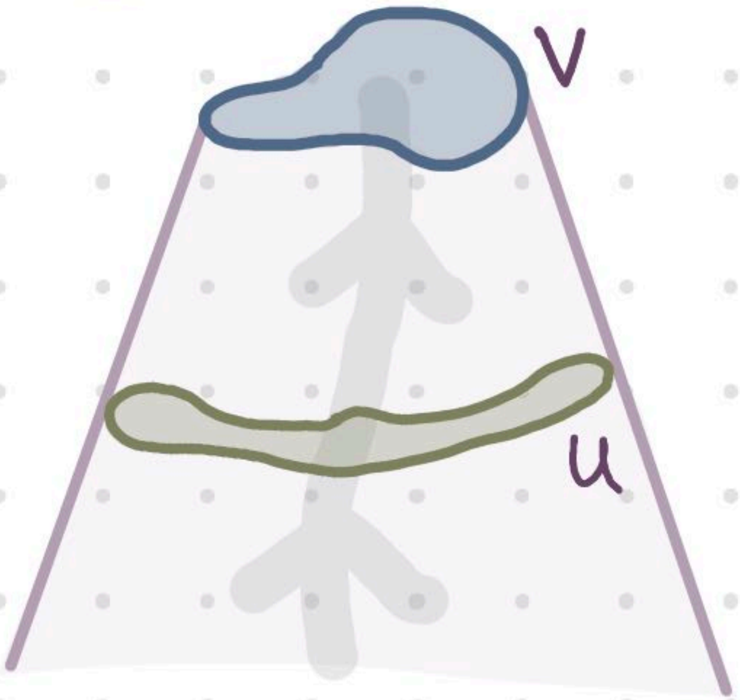
IDEA

all info. reaching V must pass through U:

$$U \in \mathrm{Cov}^-(V)$$

Causal Coverage

[Christensen, Crane 05]



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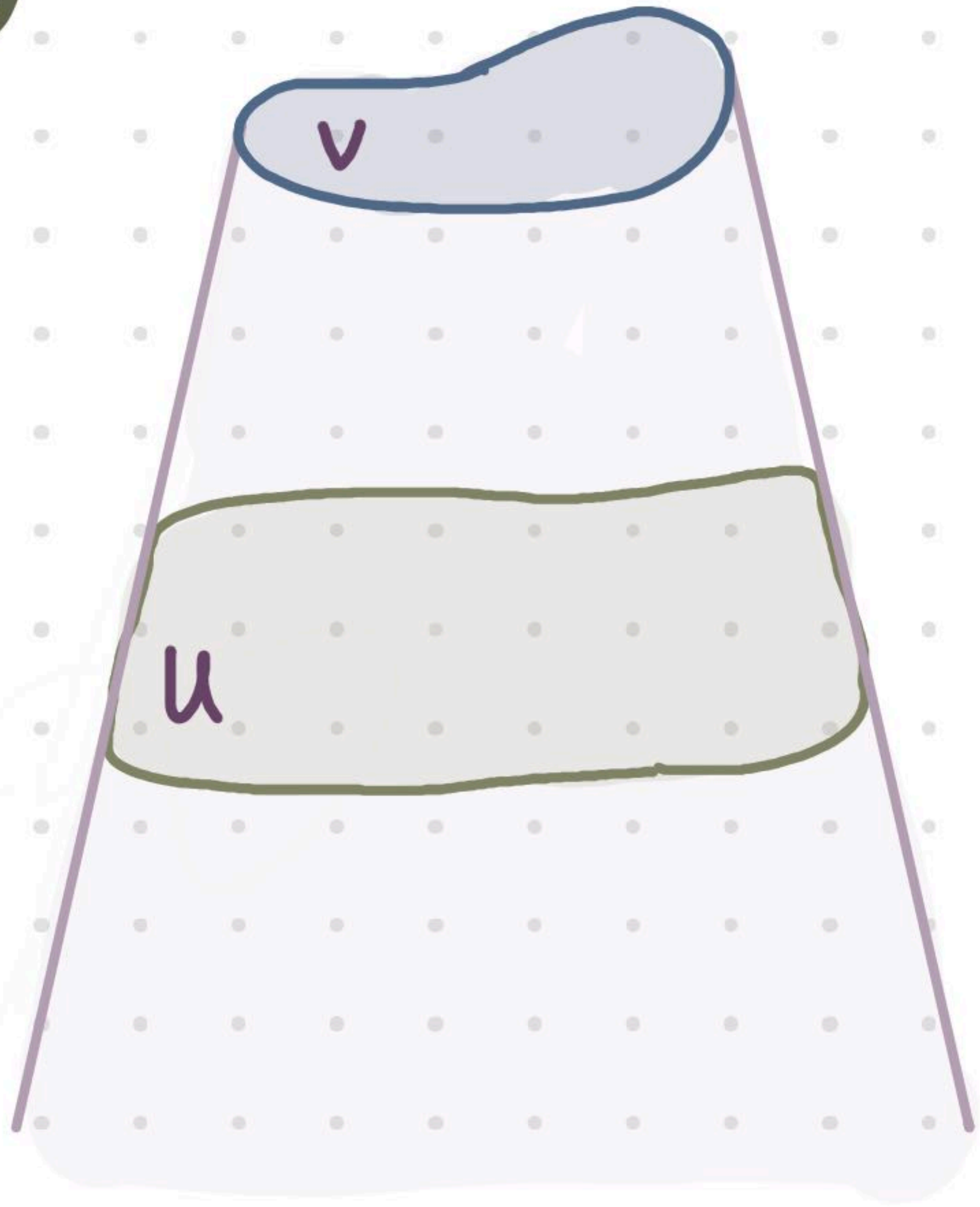

Causal Coverage

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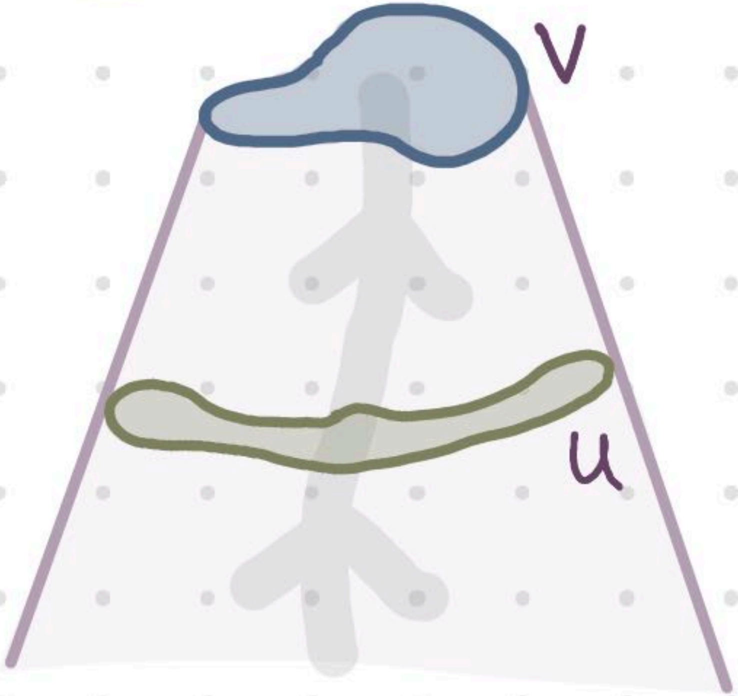
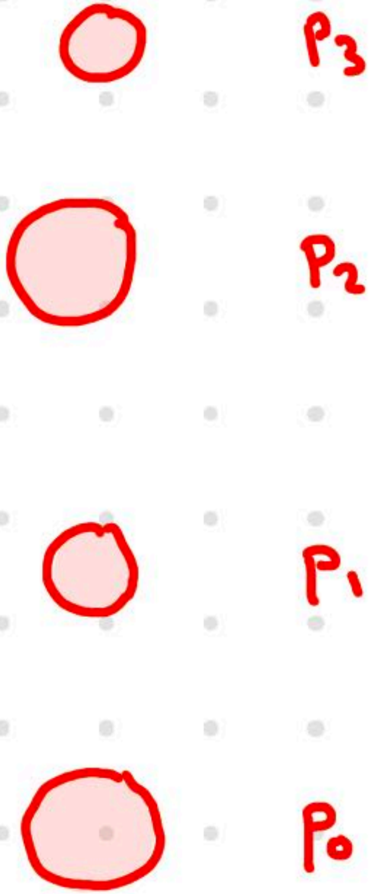


PATHS a path is a finite chain:

$$p_0 \trianglelefteq p_1 \trianglelefteq \cdots \trianglelefteq p_{N-1} \trianglelefteq p_N$$

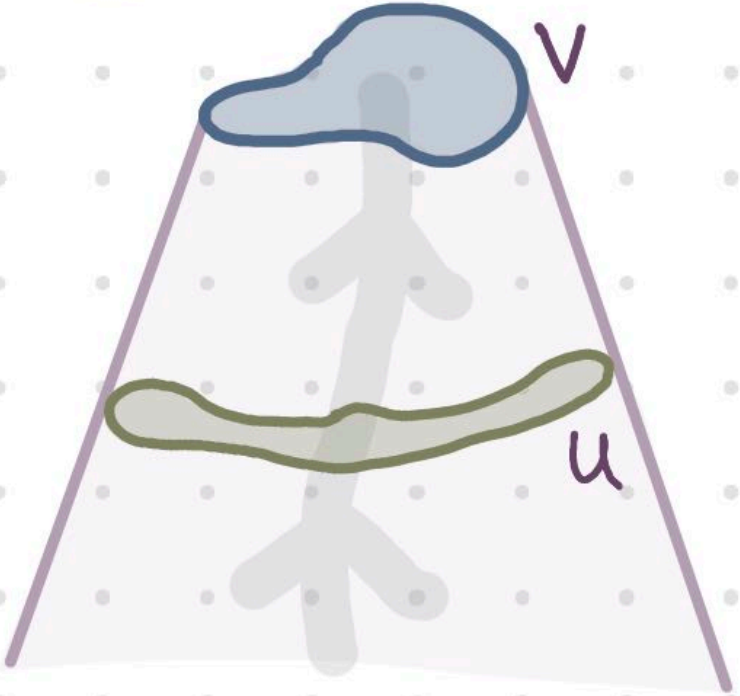


IDEA all info. reaching V must pass through U:

$$U \in Cov^{-}(V)$$

Causal Coverage

[Christensen, Crane 05]



PATHS

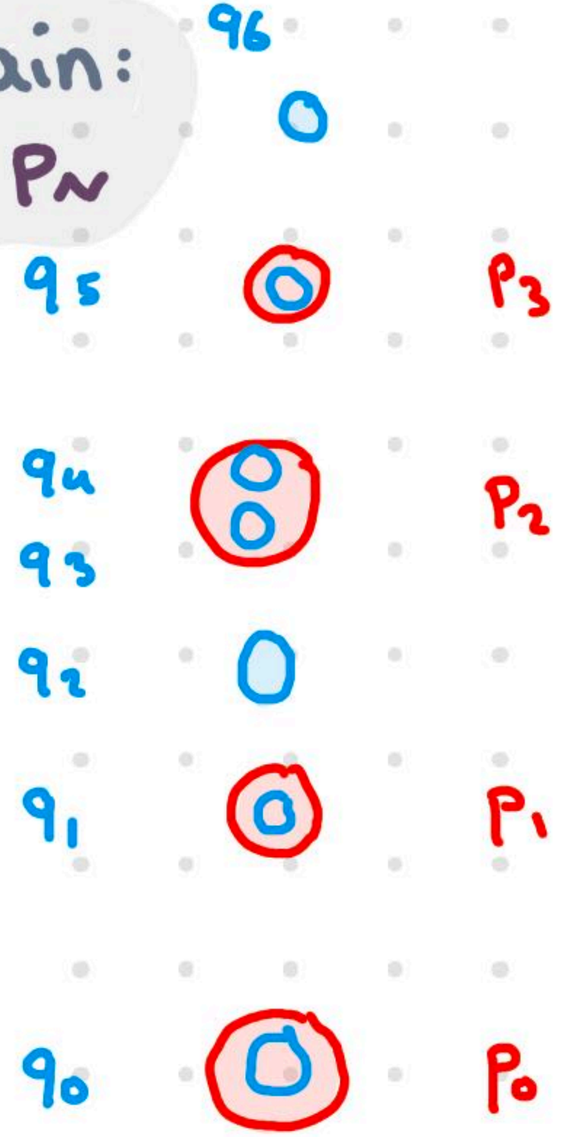
a path is a finite chain:

$$p_0 \trianglelefteq p_1 \trianglelefteq \cdots \trianglelefteq p_{N-1} \trianglelefteq p_N$$

REFINES

q refines p if:

$$\forall n \exists m : q_m \sqsubseteq p_n .$$



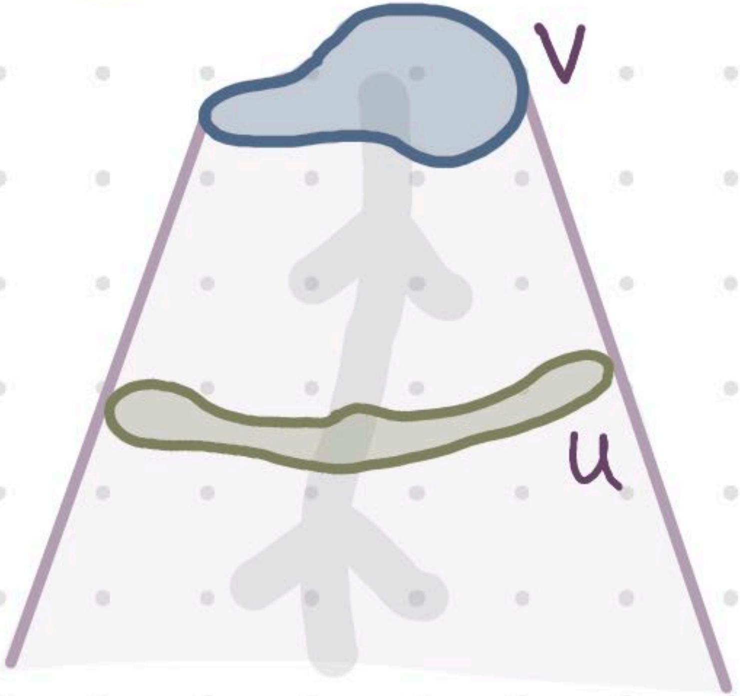
IDEA

all info. reaching V **must** pass through U:

$$U \in \mathrm{Cov}^-(V)$$

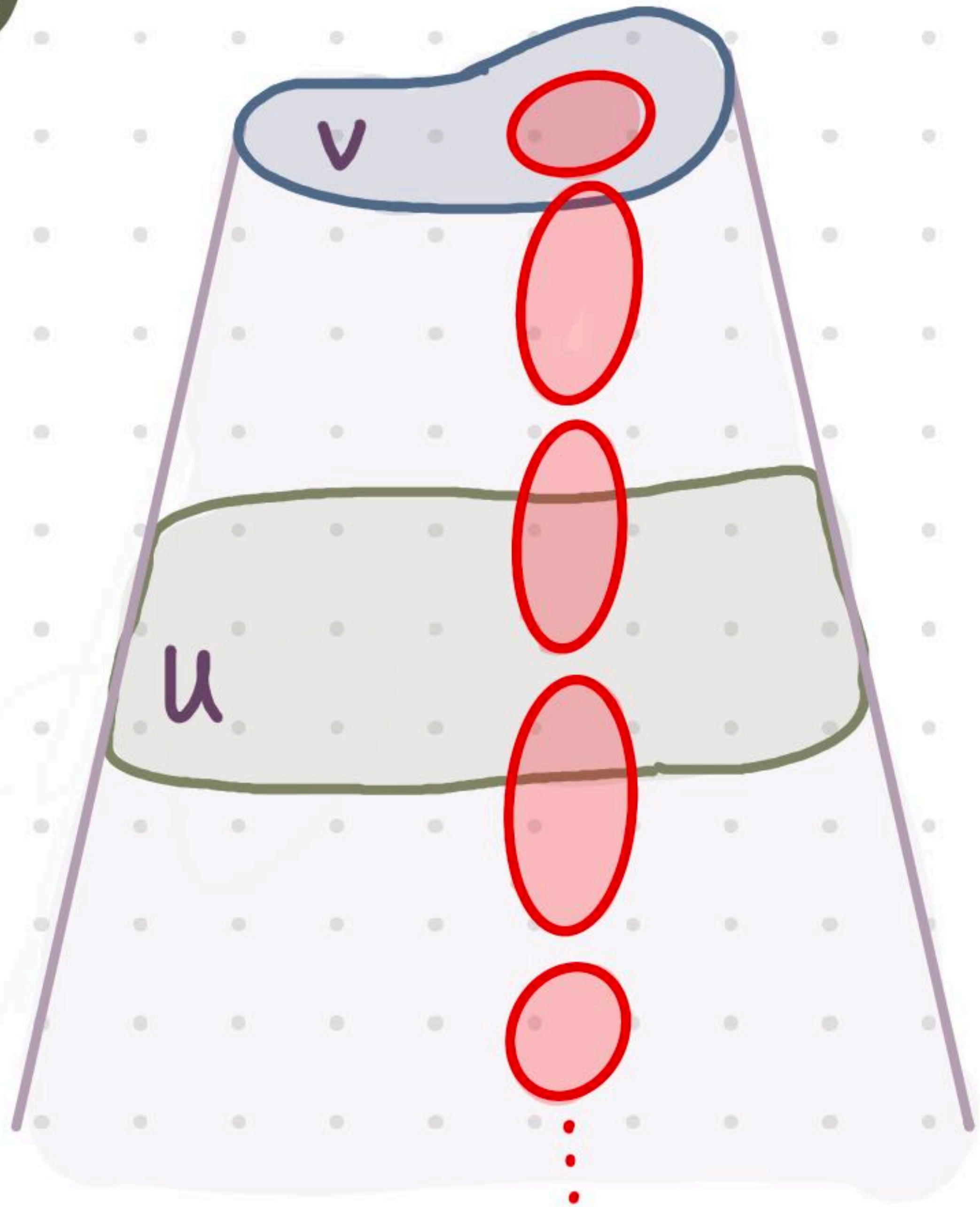
Causal Coverage

[Christensen, Crane 05]



IDEA: all info. reaching V must pass through U:

$U \in \mathrm{Cov}^-(V)$



Causal Coverage

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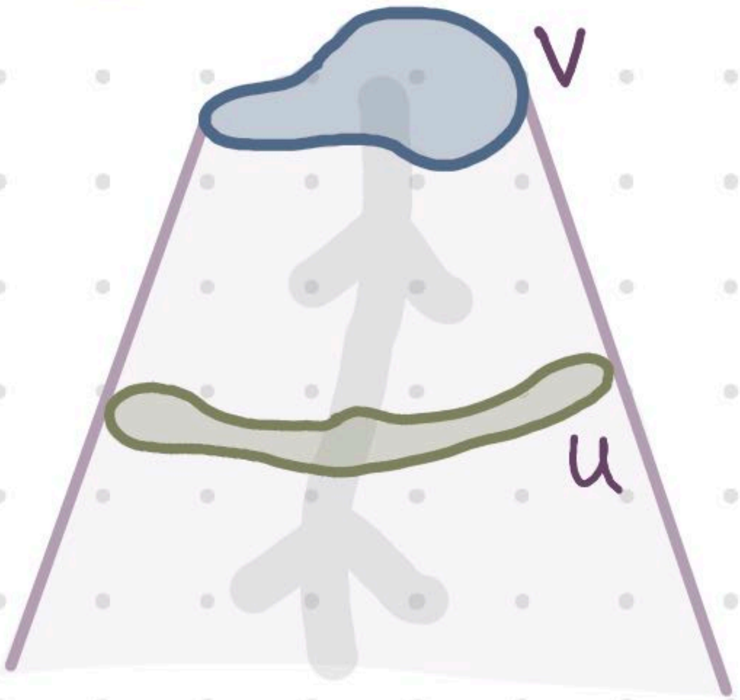

IDEA: all info. reaching V must pass through U:

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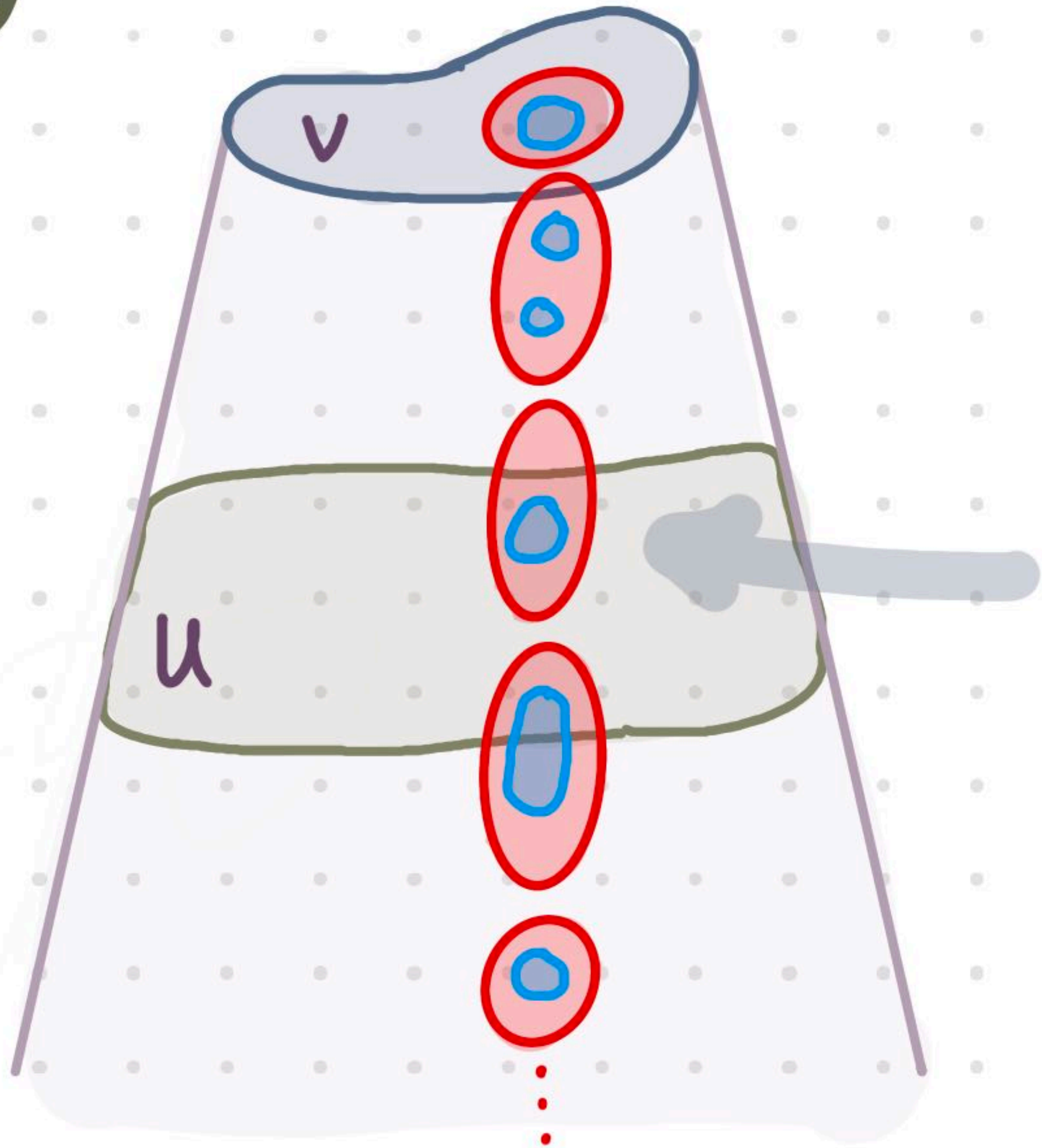

Causal Coverage

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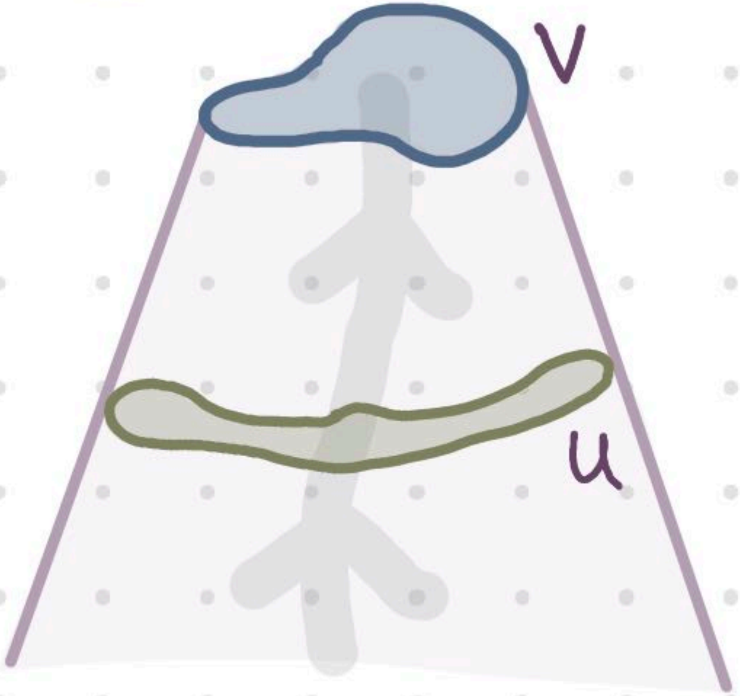
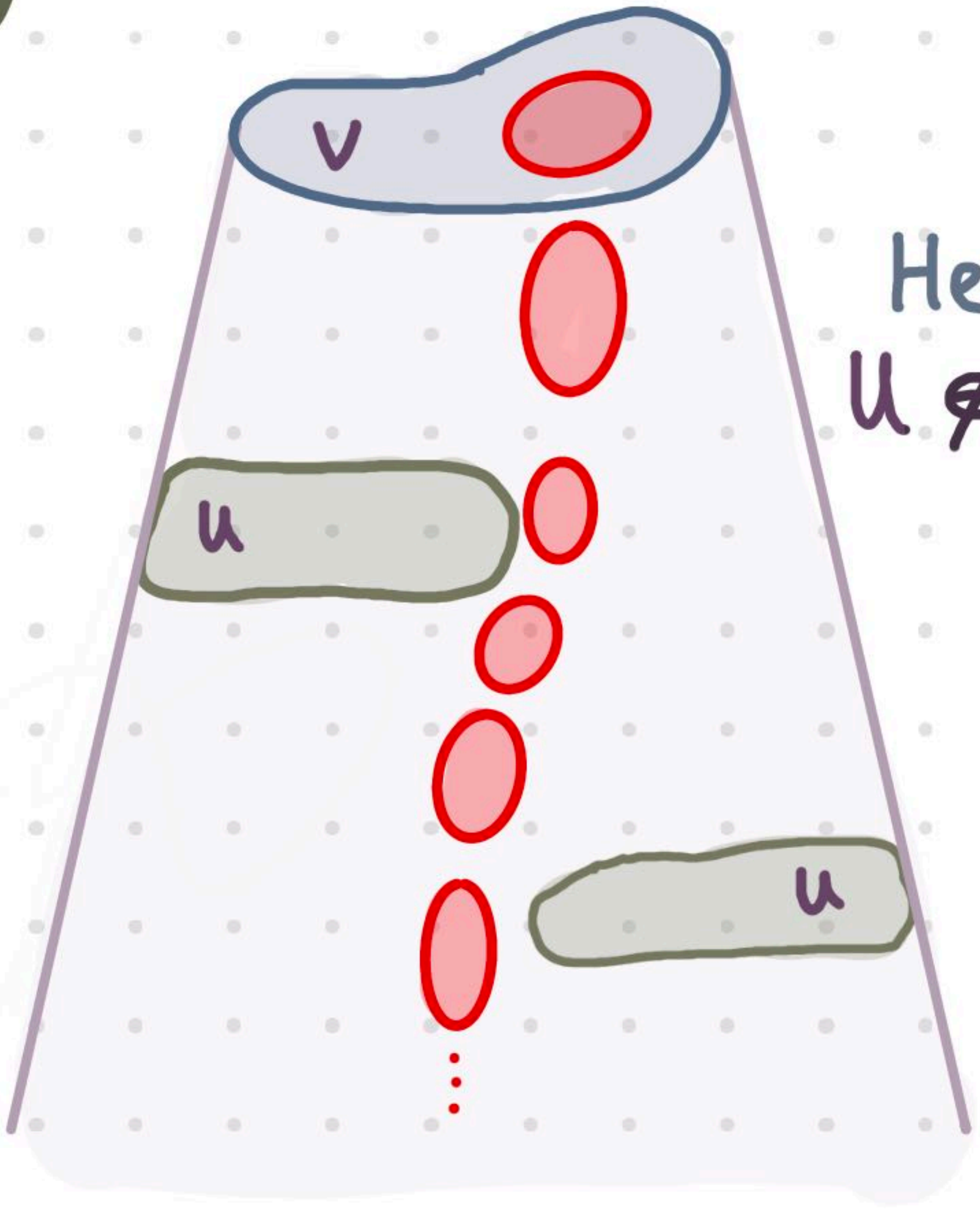


IDEA all info. reaching V **must** pass through U:

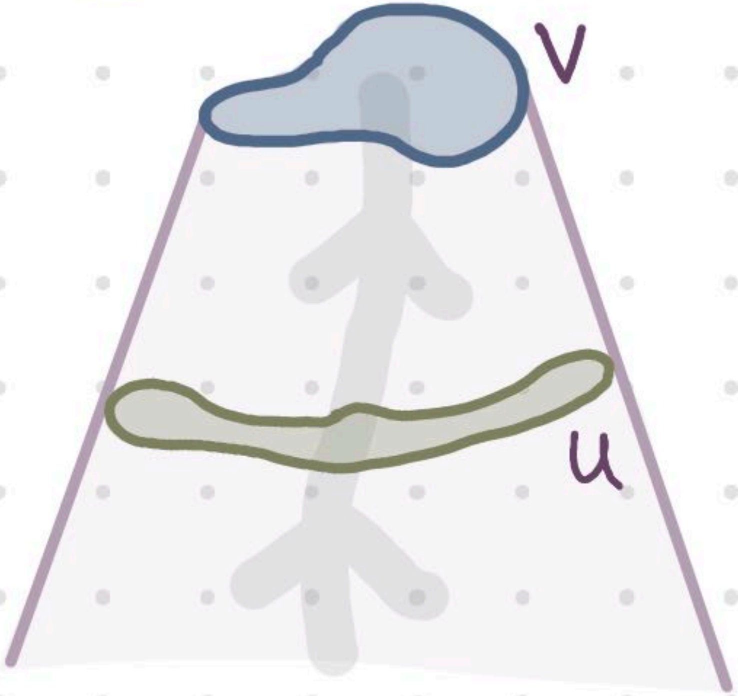
$$U \in \mathrm{Cov}^{-}(V)$$



Here: $U \notin \mathrm{Cov}^{-}(V)$

Causal Coverage

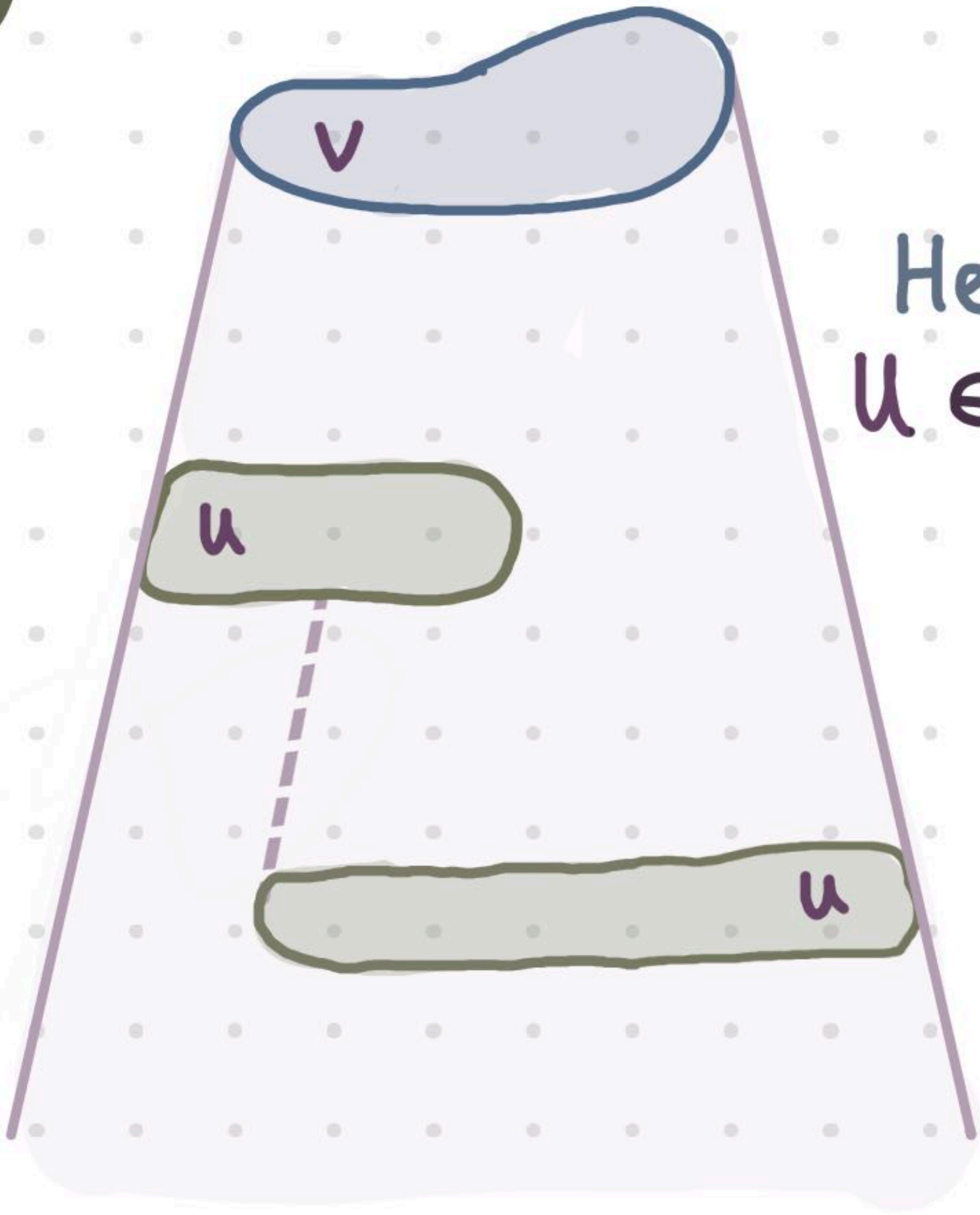
[Christensen, Crane 05]



IDEA

all info. reaching v **must** pass through u:

$$u \in \mathrm{Cov}^-(v)$$



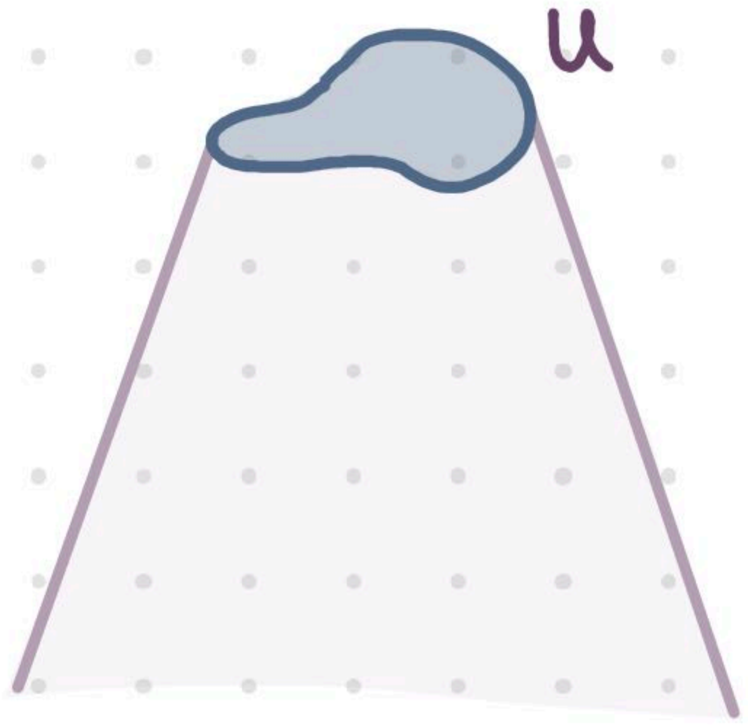
Here:

$u \in \mathrm{Cov}^-(v)$

Causal Coverage

LEMMA

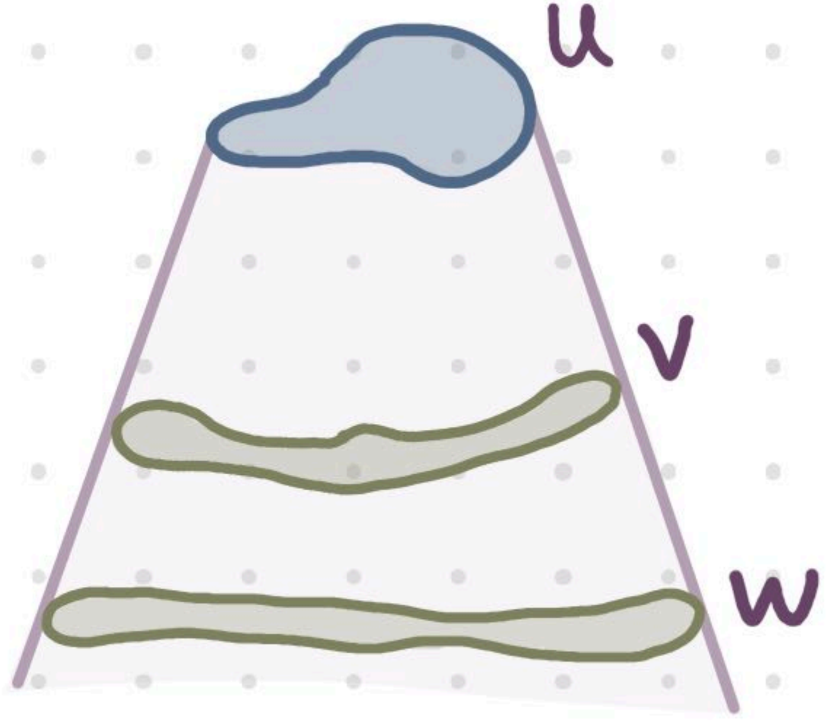
- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$



Causal Coverage

LEMMA

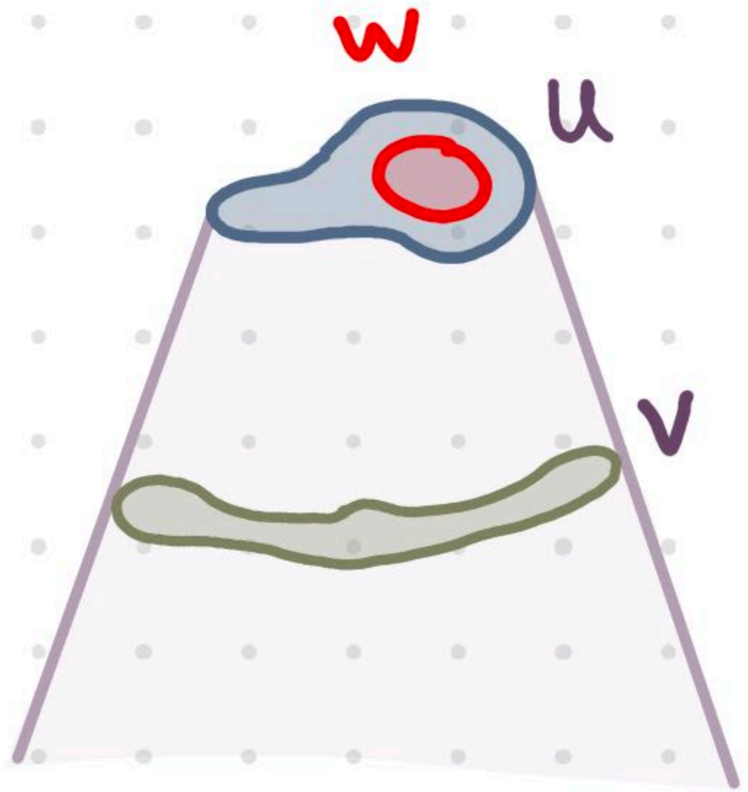
- $U \in Cov^-(U)$
- $\downarrow U \in Cov^-(U)$
- $W \in Cov^-(V),\ V \in Cov^-(U)$
  $\implies W \in Cov^-(U)$



Causal Coverage

LEMMA

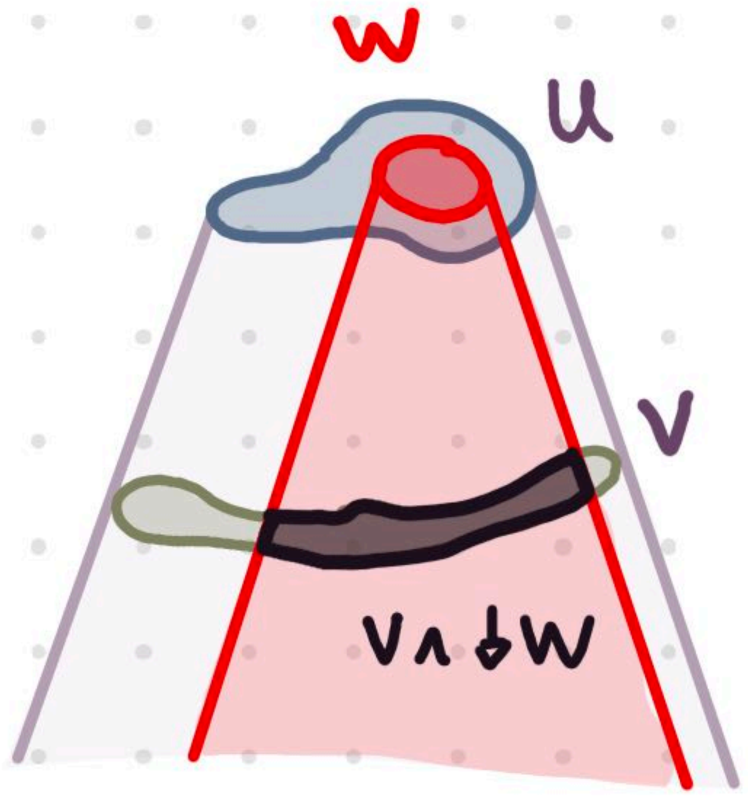
- $U \in Cov^-(U)$
- $\downarrow U \in Cov^-(U)$
- $W \in Cov^-(V),\ V \in Cov^-(U)$
  $\implies W \in Cov^-(U)$
- $V \in Cov^-(U),\ W \subseteq U$
  $\implies V \wedge \downarrow W \in Cov^-(W)$



Causal Coverage

LEMMA

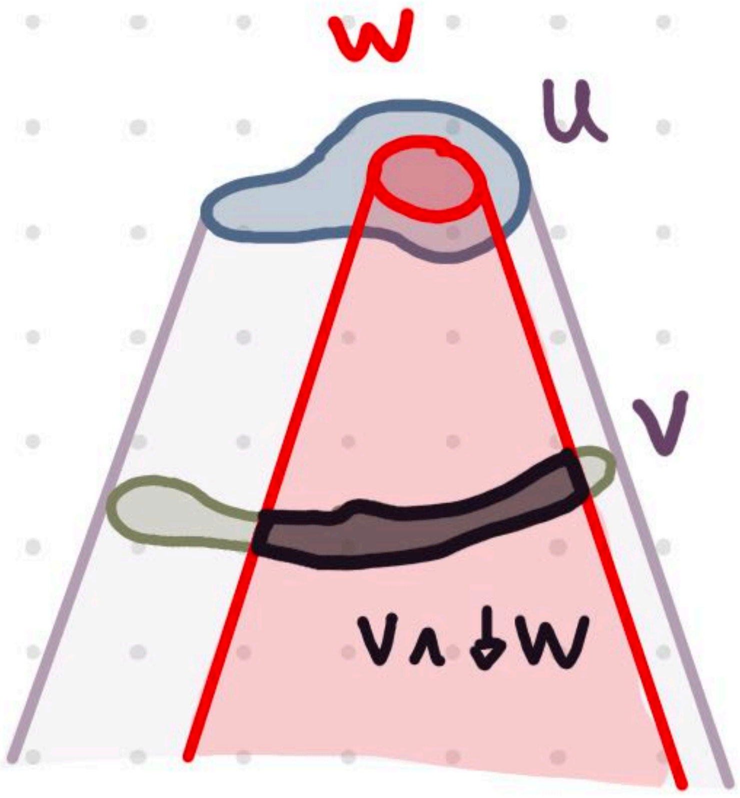
- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V),\ V \in \mathrm{Cov}^-(U)$
  $\Longrightarrow\ W \in \mathrm{Cov}^-(U)$
- $V \in \mathrm{Cov}^-(U),\ W \subseteq U$
  $\Longrightarrow\ V \wedge \downarrow W \in \mathrm{Cov}^-(W)$



Causal Coverage

LEMMA

- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V)$, $V \in \mathrm{Cov}^-(U)$
  $\Longrightarrow W \in \mathrm{Cov}^-(U)$
- $V \in \mathrm{Cov}^-(U)$, $W \sqsubseteq U$
  $\Longrightarrow V \wedge \downarrow W \in \mathrm{Cov}^-(W)$



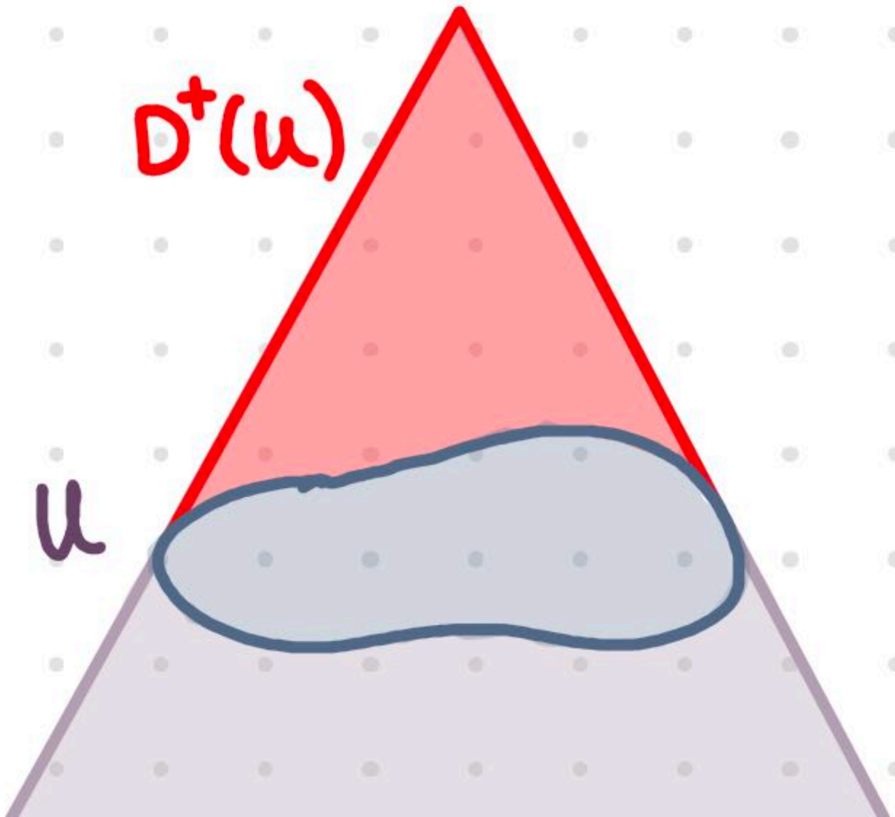
Grothendieck topology on $\mathcal{K}\ell(\downarrow)$

Causal Coverage

[Geroch 70]

DOMAIN OF DEPENDENCE

$$D^+(U) := \bigvee \{ W \in \mathcal{O}X : U \in \mathrm{Cov}^-(W) \}$$



EXAMPLE

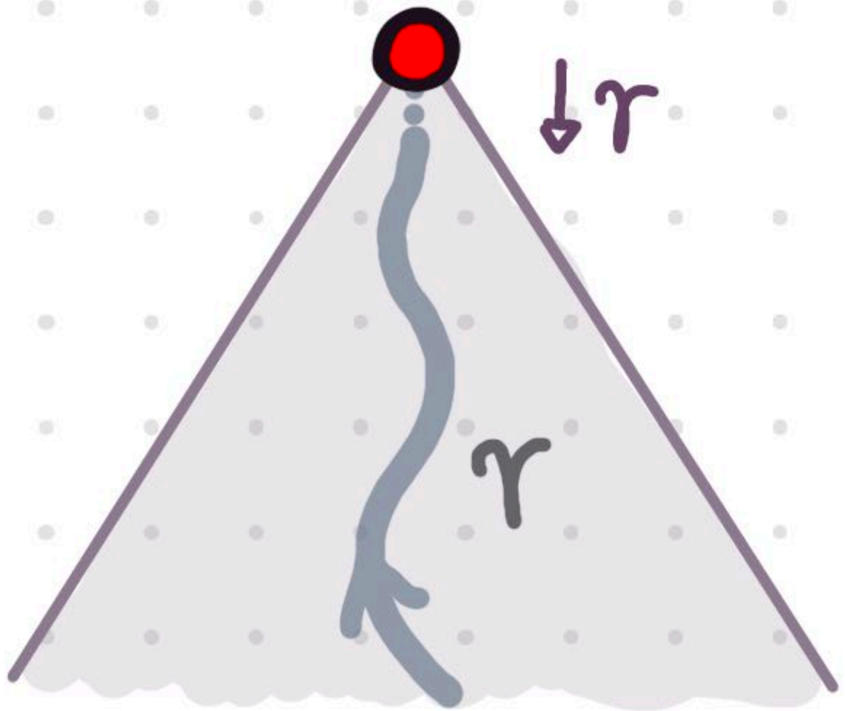
Sheaf of solutions to the wave equation on Minkowski space

Q. $\mathrm{Sh}(X, \trianglelefteq)$? Internal logic related to modal logic approaches of spacetime ? [Goldblatt 80, 92]

Causal Boundaries

joint with:
Prakash Panangaden

IDEA add ideal points to space(time) via would-be limits of curves γ



THEOREM $P \in \mathrm{im}(\downarrow)$ is coprime

iff

$\exists$ causal curve γ s.t. $P = \downarrow\gamma$

[Geroch, Kronheimer, Penrose 72]

Causal Boundaries

IDEA

add ideal points to space(time) via would-be limits of curves γ

$\mathrm{im}(\uparrow)$ is a frame, so defines locale M^Δ.

THEOREM

$P \in \mathrm{im}(\downarrow)$ is coprime iff $\exists$ causal curve γ s.t. $P = \downarrow\gamma$

[Geroch, Kronheimer, Penrose 72]

LEMMA

there is a bijection $IP(M) \cong pt(M^\Delta)$

$$\begin{array}{ccc} IP(M) & \cong & pt(M^\Delta) \\ \wr\wr & & \\ IP(M_{\neg\neg}) & \cong & pt(M_{\neg\neg}^\Delta) \end{array}$$

Causal Boundaries

LEMMA There is an isomorphism:

$$\left\{\begin{matrix}\text{"convex" ordered} \\ \text{locales s.t. } \uparrow, \downarrow \\ \text{join-preserving}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{"reflective"} \\ \text{biframes}\end{matrix}\right\}^{op}$$

$$(X, \trianglelefteq) \longmapsto (\mathcal{O}X, \mathrm{im}(\uparrow), \mathrm{im}(\downarrow))$$

Q. biframe compactifications for ordered locales?