Relativistic concepts in point-free spaces

Oxford Quantum Lunch

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joint with
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The University of Edinburgh
School of Informatics

Overview

Ordered Locales

Causal Coverage

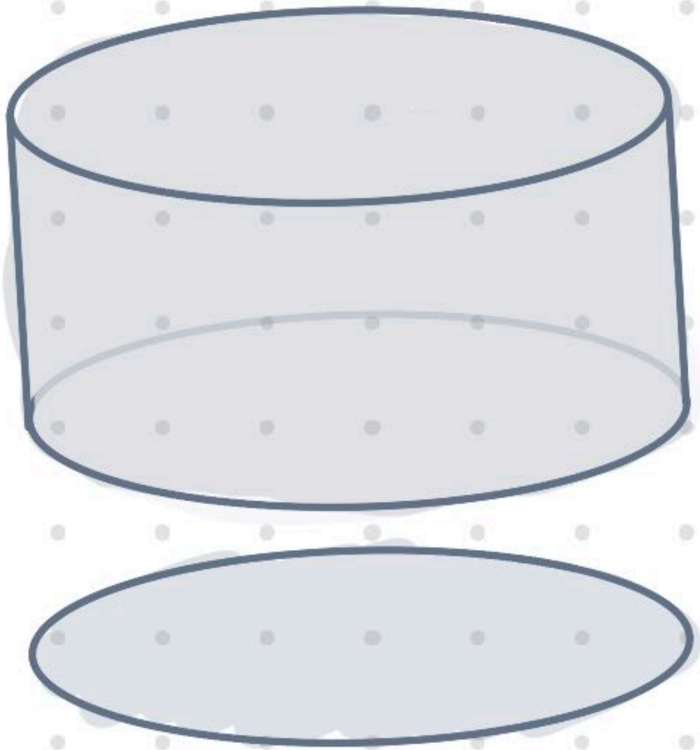
Causal Boundaries

(more) Future Work

Motivation

Spacetimes e.g. [Forrest 96], [Isham 90], [Sorkin 91]

tensor topology [Moliner, Heunen, Tull 20], [Soares Barbosa, Heunen 23]



C

ZI(C)

EXAMPLES

$$ZI(\mathrm{Sh}(S)) \cong \mathcal{O}S$$

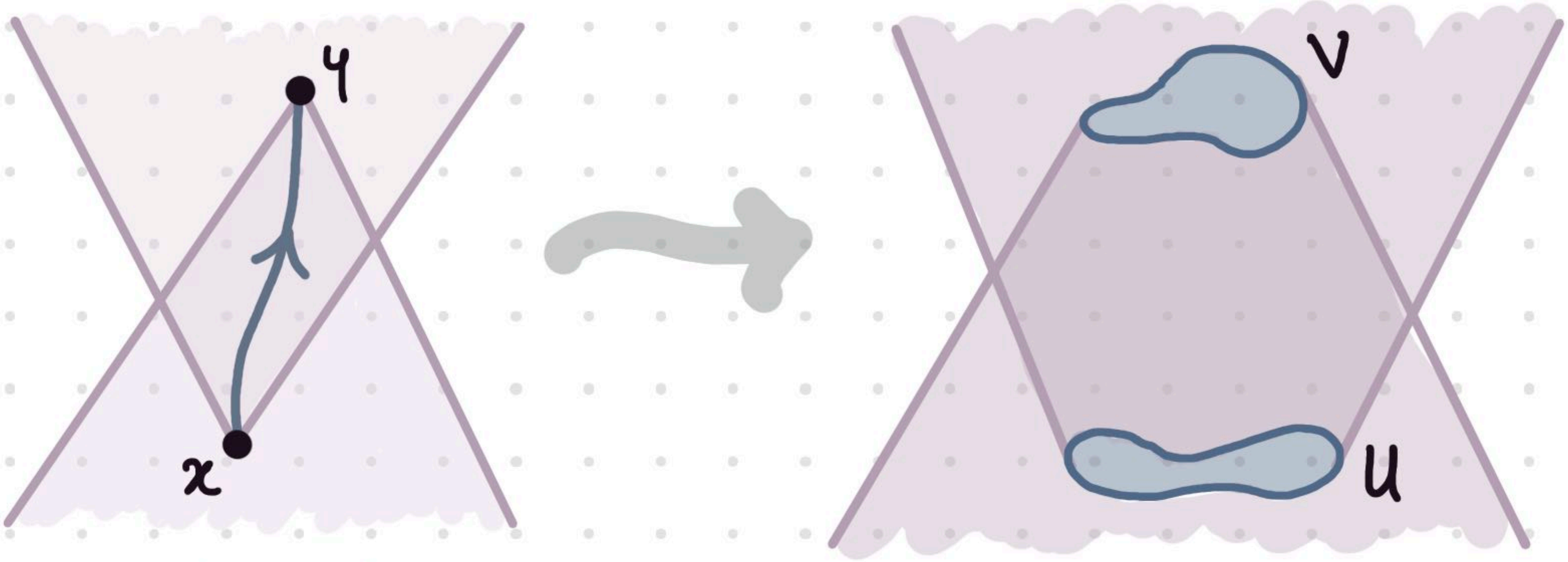
$$ZI(\mathrm{Hilb}_{C_0(S)}) \cong \mathcal{O}S$$

Idea

$$\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[u, hook] \arrow[l, shift left, "\mathrm{pt}"]
\end{tikzcd}$$

Causality + spaces

Idea



causal order

$x \leq y$

region causality

$U \trianglelefteq V$

$$
\begin{tikzcd}
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\mathrm{Top} \arrow[u, hook] \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[u, hook] \arrow[l, shift left=2]
\end{tikzcd}
$$

$$\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left=1.5] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[u, hook] \arrow[l, shift left=1.5]
\end{tikzcd}$$

Locales

$$\begin{array}{ccc} \mathbf{Top} & \longrightarrow & \mathbf{Frm}^{op} \\ S & \longmapsto & \mathcal{O}S \\ (S \xrightarrow{f} T) & \longmapsto & (\mathcal{O}T \xrightarrow{f^{-1}} \mathcal{O}S) \end{array}$$

Frames as algebraic dual to a type of space:

$$\mathbf{Loc} := (\mathbf{Frm})^{op}$$

LOCALES

object: $X \in \mathbf{Loc} \longleftrightarrow \mathcal{O}X \in \mathbf{Frm}$

arrows: $(X \xrightarrow{f} Y) \in \mathbf{Loc} \longleftrightarrow (\mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X) \in \mathbf{Frm}$

Locales

Not every locale comes from a topological space!! Yet:

POINT

map of locales:

$p : 1 \longrightarrow X$

$$\mathcal{O}X \xrightarrow{p^{-1}} \mathcal{O}1 := \{F < T\}$$

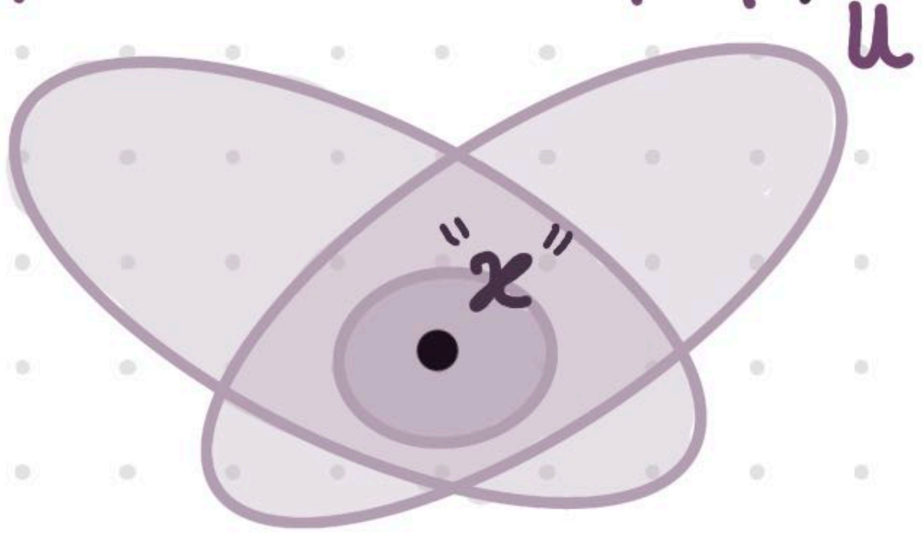
$$\Updownarrow$$

$$\mathcal{F} = \{U \in \mathcal{O}X : p^{-1}(U) = T\}$$

$$\mathbf{Loc} \xrightarrow{pt} \mathbf{Top}$$

$$X \longmapsto pt(X)$$

$$f \longmapsto pt(f)$$



$$U \in \mathcal{F} \iff \text{"}x\text{"} \in U$$

Locales

Not every locale comes from a topological space!! Yet:

$$\begin{array}{ccc} \mathbf{Loc} & \xrightarrow{pt} & \mathbf{Top} \\ X & \longmapsto & pt(X) \\ f & \longmapsto & pt(f) \end{array}$$

u

POINT

map of locales:

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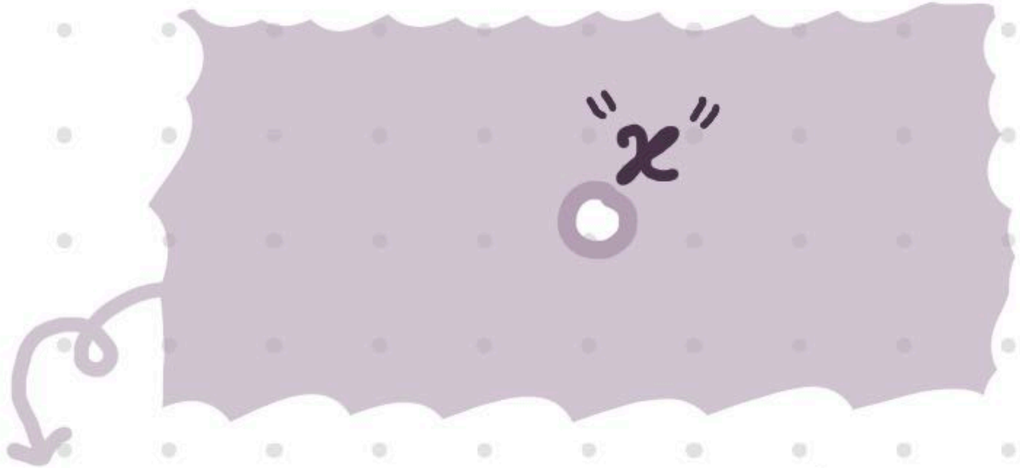


$$V(\mathcal{O}X \setminus \mathcal{F}) \sim X \setminus \overline{\{\text{"}x\text{"}\}}$$

Locales

THEOREM

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left=2, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, shift left=2, "\mathrm{pt}"] \end{tikzcd}$$

Locales

Start with $S \in \mathbf{Top}$.

REG. $\mathcal{O}$. $$\mathcal{R}S := \{ U \in \mathcal{O}S \ : \ U = (\overline{U})^{\circ} \}$$ defines $\mathrm{Reg}(S) \in \mathbf{Loc}$.

If S is Hausdorff:

LEMMA $$\mathrm{Pt}\,\mathrm{Reg}(S) \cong \{ \text{isolated points in } S \}$$

E.G. $\mathrm{Pt}\,\mathrm{Reg}(\mathbb{R}^n) = \emptyset$

$$\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left=2] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[l, shift left=2] \arrow[u, hook]
\end{tikzcd}$$

Ordered Spaces

ORD. SP. A space S with a preorder $\leqslant$.

MAPS continuous monotone functions $f : S \longrightarrow T$.

$$x \leqslant y \implies f(x) \leqslant f(y).$$

We get a category:

OrdTop

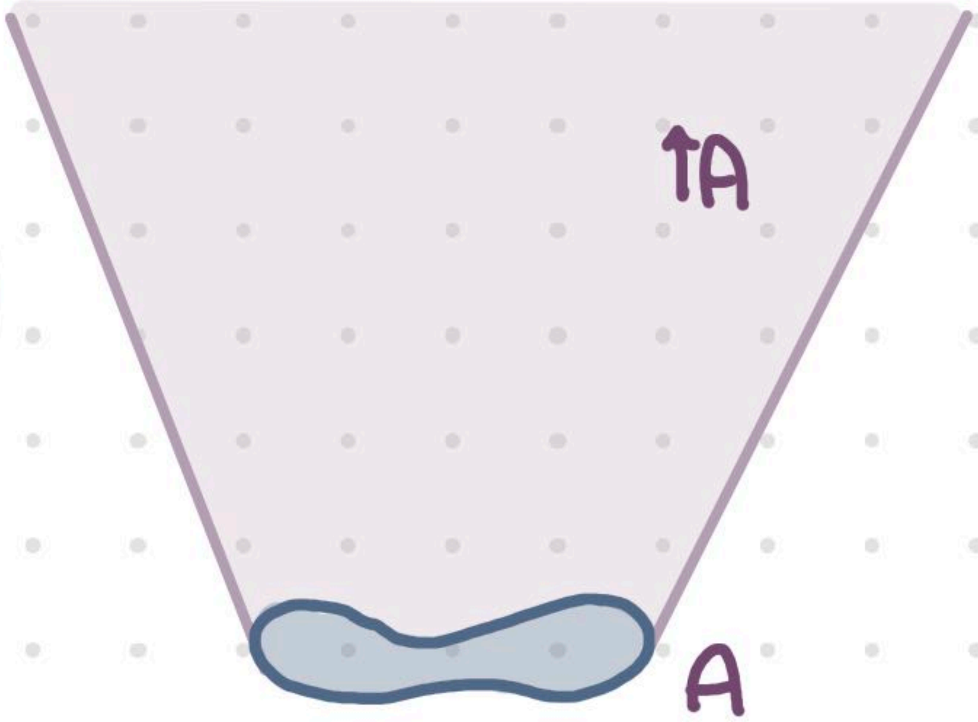
Ordered Spaces

capture $\leqslant$ in terms of open regions?

CONES

the future cone of $A \subseteq S$:

$$\uparrow A := \{ x \in S \mid \exists a \in A : a \leqslant x \}.$$



LEMMA

- $A \subseteq B \implies \uparrow A \subseteq \uparrow B$
- $A \subseteq \uparrow A$
- $\uparrow\uparrow A \subseteq \uparrow A$

$$x \leqslant y \iff y \in \uparrow\{x\}$$

Ordered Spaces

capture $\leqslant$ in terms of open regions?

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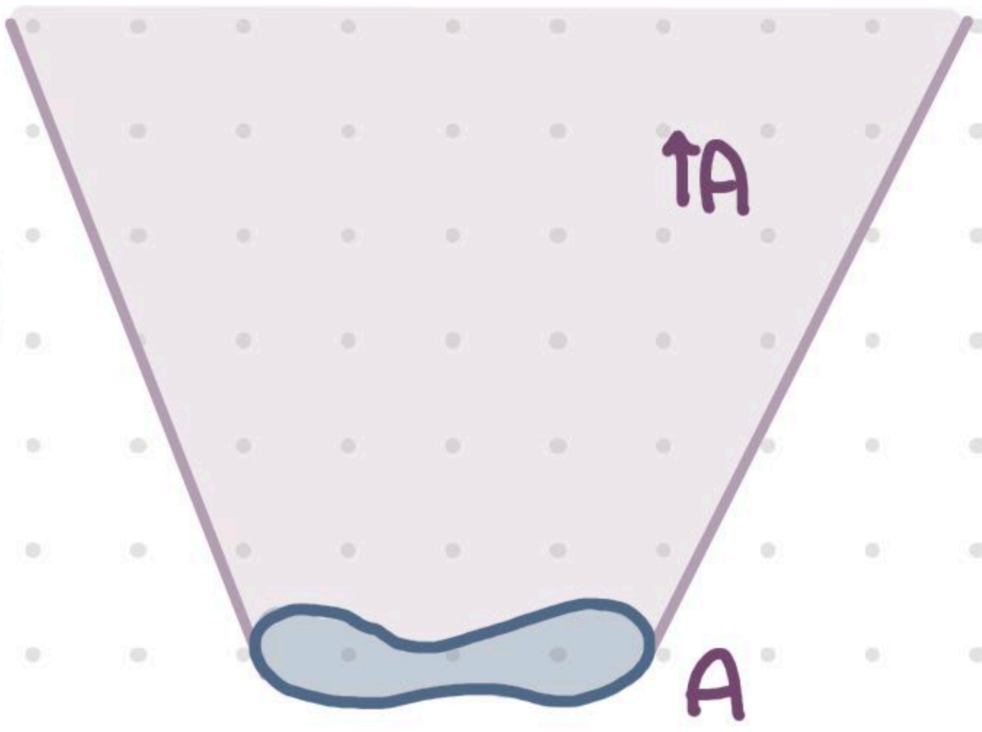


LEMMA

$f : S \longrightarrow T$ monotone iff:

$$\uparrow f^{-1}(B) \subseteq f^{-1}(\uparrow B).$$

Ordered Spaces

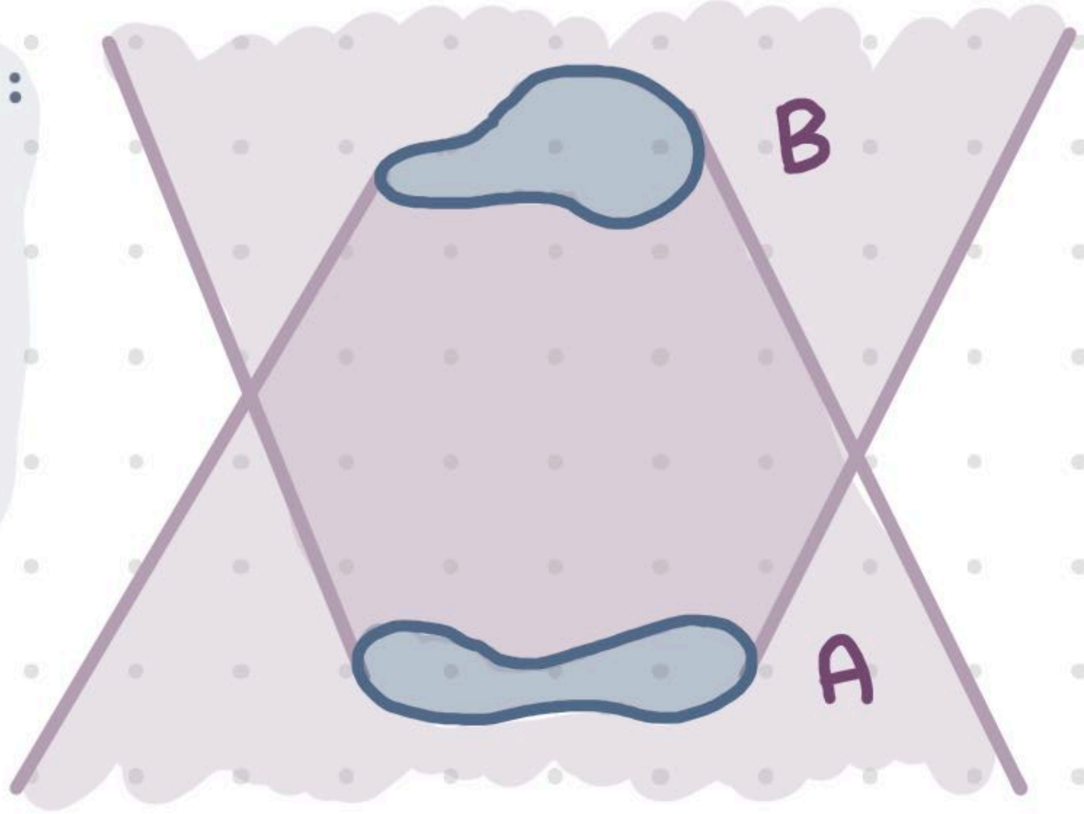
capture $\leqslant$ in terms of open regions?

EM ORDER

If $(S, \leqslant)$ is a preorder:

$$A \trianglelefteq B \iff \begin{cases} A \subseteq {\downarrow} B, \\ B \subseteq {\uparrow} A \end{cases}$$

is a preorder on $\mathcal{P}(S)$.



$(\mathrm{Loc}(S), \trianglelefteq)$ as prototypical ordered locale

$$
\begin{tikzcd}
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\mathrm{Top} \arrow[u, hook] \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[l, shift left=2] \arrow[u, hook]
\end{tikzcd}
$$

Ordered Locales

ORD. LOC.

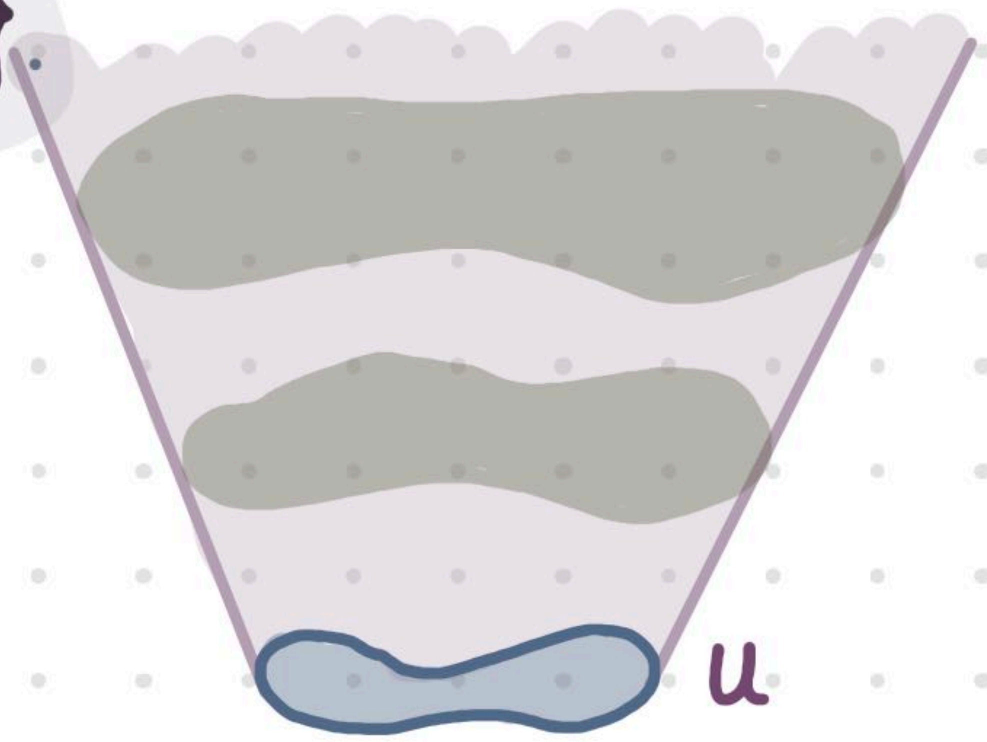
a locale X with preorder $\trianglelefteq$ on $\mathcal{O}X$:

$$\forall i: U_i \trianglelefteq V_i \implies \bigvee U_i \trianglelefteq \bigvee V_i.$$

Ordered Locales

CONES

$$\uparrow U := \bigvee \{ W \in \mathcal{O}X : U \trianglelefteq W \}.$$



LEMMA

- $\downarrow U \trianglelefteq U \trianglelefteq \uparrow U$
- $U \subseteq V \implies \uparrow U \subseteq \uparrow V$
- $U \subseteq \uparrow U$
- $\uparrow\uparrow U \subseteq \uparrow U$

$$U \trianglelefteq W,\ V \trianglelefteq V \implies V = U \vee V \trianglelefteq V \vee W$$
$$\implies W \subseteq V \vee W \subseteq \uparrow V.$$

Ordered Locales

CONES

$$\Uparrow U := \bigvee\{W \in \mathcal{O}X : U \trianglelefteq W\}.$$

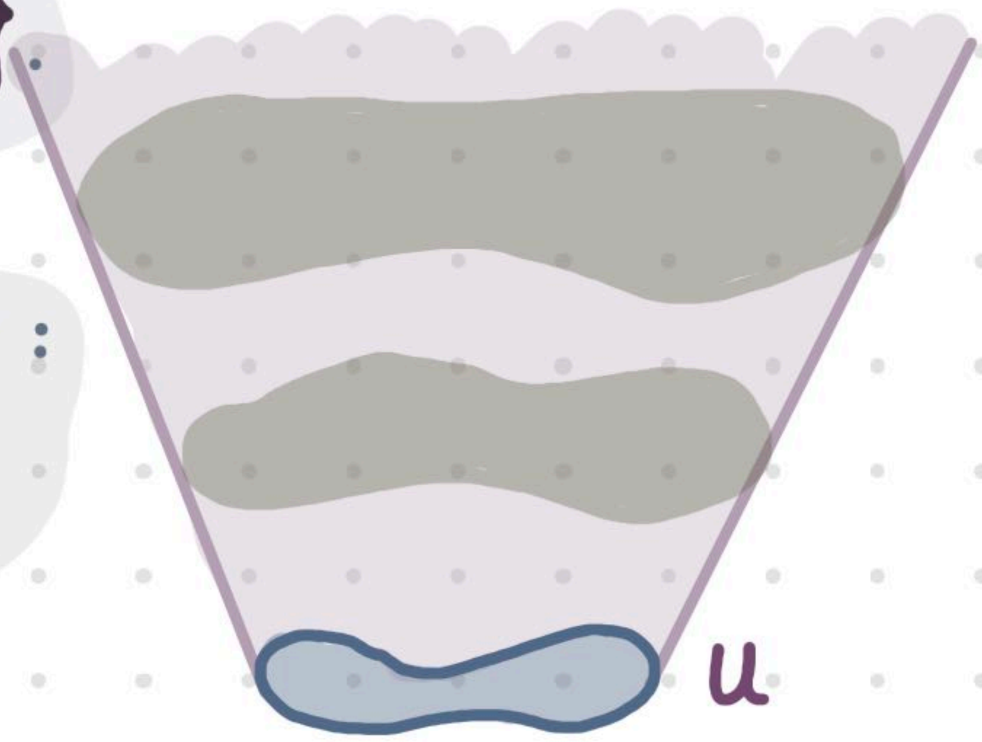
MAPS

monotone map $f : X \longrightarrow Y$:

$$\Uparrow f^{-1}(U) \subseteq f^{-1}(\Uparrow U).$$

We get a category:

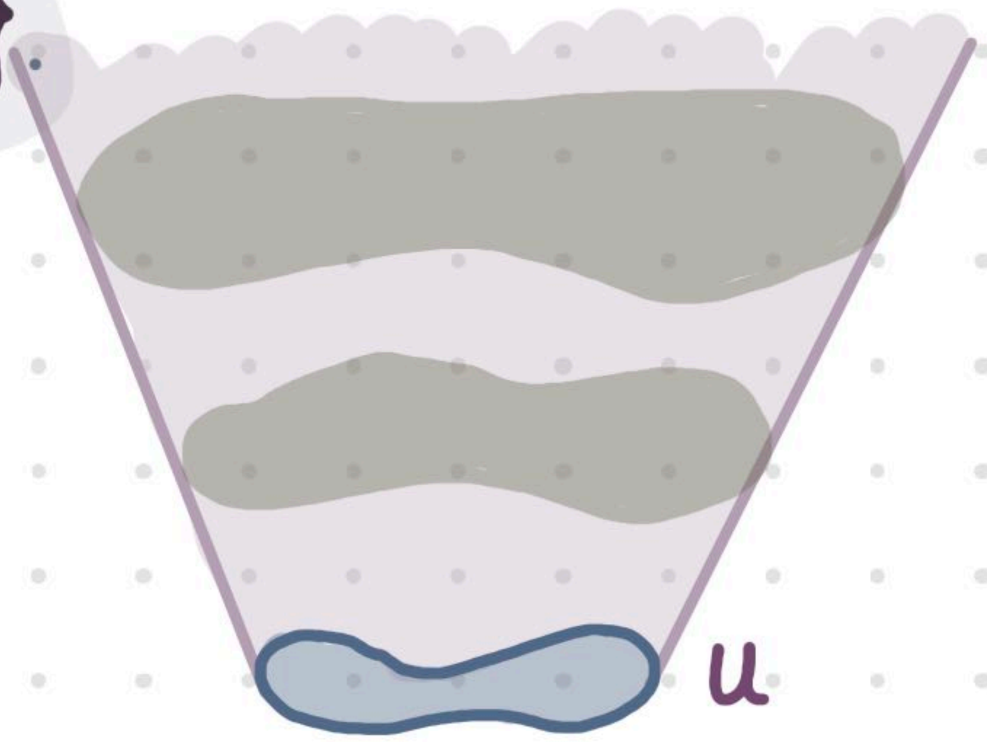
OrdLoc



Ordered Locales

CONES

$$\Uparrow U := \bigvee \{ W \in \mathcal{O}X : U \triangleleft W \}.$$



E.G. in a space $(S, \leq)$:

$$\Uparrow U = (\uparrow U)^\circ, \quad \Downarrow U = (\downarrow U)^\circ$$

In spacetime:

$$\Uparrow U = I^+(U) = J^+(U),$$
$$\Downarrow U = I^-(U) = J^-(U)$$

Adjunction

$$\begin{tikzcd}
\mathbf{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left] \\
(S, \leqslant) \arrow[r, mapsto] & (\mathrm{Loc}(S), ?) \\
(\mathrm{pt}(X), ?) & (X, \trianglelefteq) \arrow[l, mapsto]
\end{tikzcd}$$

Adjunction

$$\begin{tikzcd} \mathbf{OrdTop} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left=1.5ex] \end{tikzcd}$$

$$(S, \leqslant) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$$

$$(\mathrm{pt}(X), ?) \longmapsfrom (X, \trianglelefteq)$$

Egli-Milner

Adjunction

$$\mathbf{OrdTop} \longrightarrow \mathbf{OrdLoc}$$
$$(S, \leqslant) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$$

OPEN CONES

$(S, \leqslant)$ has OC if:
$\forall U \in \mathcal{O}S : \uparrow U, \downarrow U \in \mathcal{O}S$

E.G.

- smooth spacetimes
- interval topology of d.latt.
- (co)discrete spaces

LEMMA

S has OC iff:
$T \xrightarrow{g} S$ monotone $\Longrightarrow$
$\mathrm{Loc}(T) \xrightarrow{\mathrm{Loc}(g)} \mathrm{Loc}(S)$ monotone

Adjunction

$$\begin{tikzcd} \mathbf{OrdTop}_{\infty} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left] \end{tikzcd}$$

$$(S, \leq) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$$

$$(\mathrm{pt}(X), ?) \longleftarrow\!\!\mapsfrom (X, \trianglelefteq)$$

Adjunction

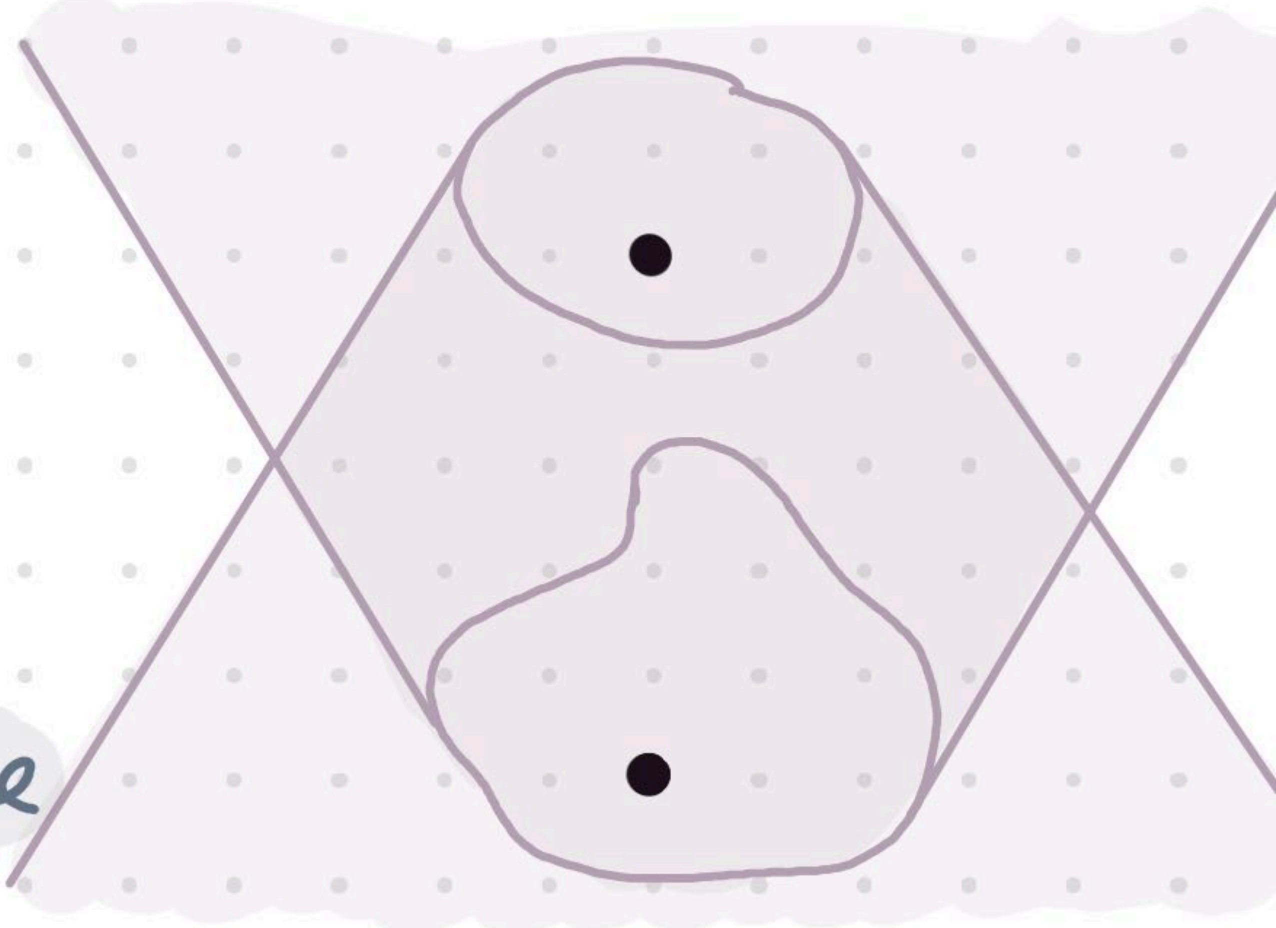
$$\mathbf{OrdLoc} \longrightarrow \mathbf{OrdTop}$$
$$(X, \trianglelefteq) \longmapsto (pt(X), \leqslant)$$

pt. ORD.

for $\mathcal{F}, \mathcal{G} \in pt(X)$:

$$\mathcal{F} \leqslant \mathcal{G} \iff \begin{array}{l} \forall U \in \mathcal{F} : \uparrow U \in \mathcal{G}, \\ \forall V \in \mathcal{G} : \downarrow V \in \mathcal{F}. \end{array}$$

$$x \leqslant y \iff \begin{array}{lll} \forall U \ni x & : & y \in \uparrow U, \\ \forall V \ni y & : & x \in \downarrow V. \end{array}$$



LEMMA

f monotone $\Rightarrow$ $pt(f)$ monotone

Adjunction

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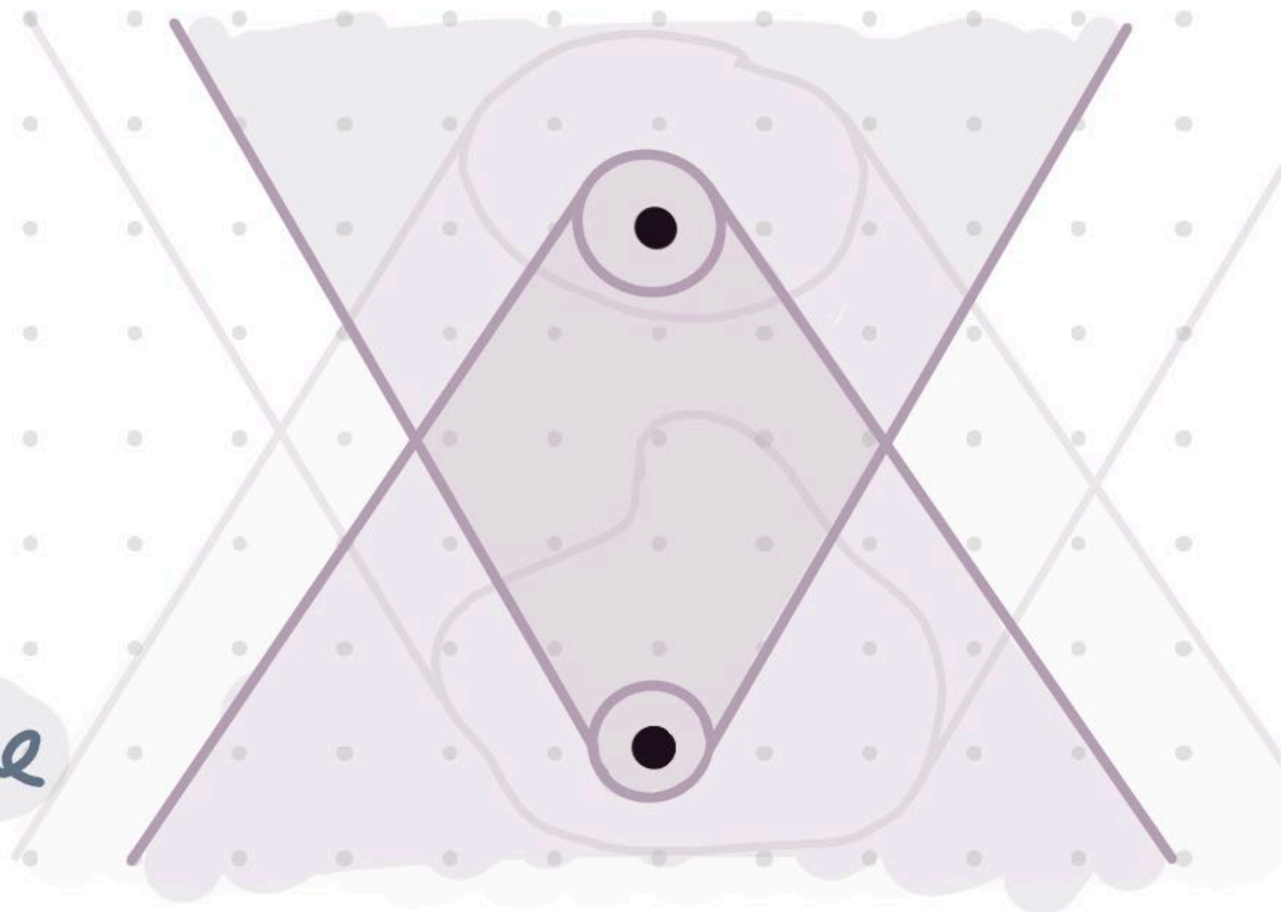

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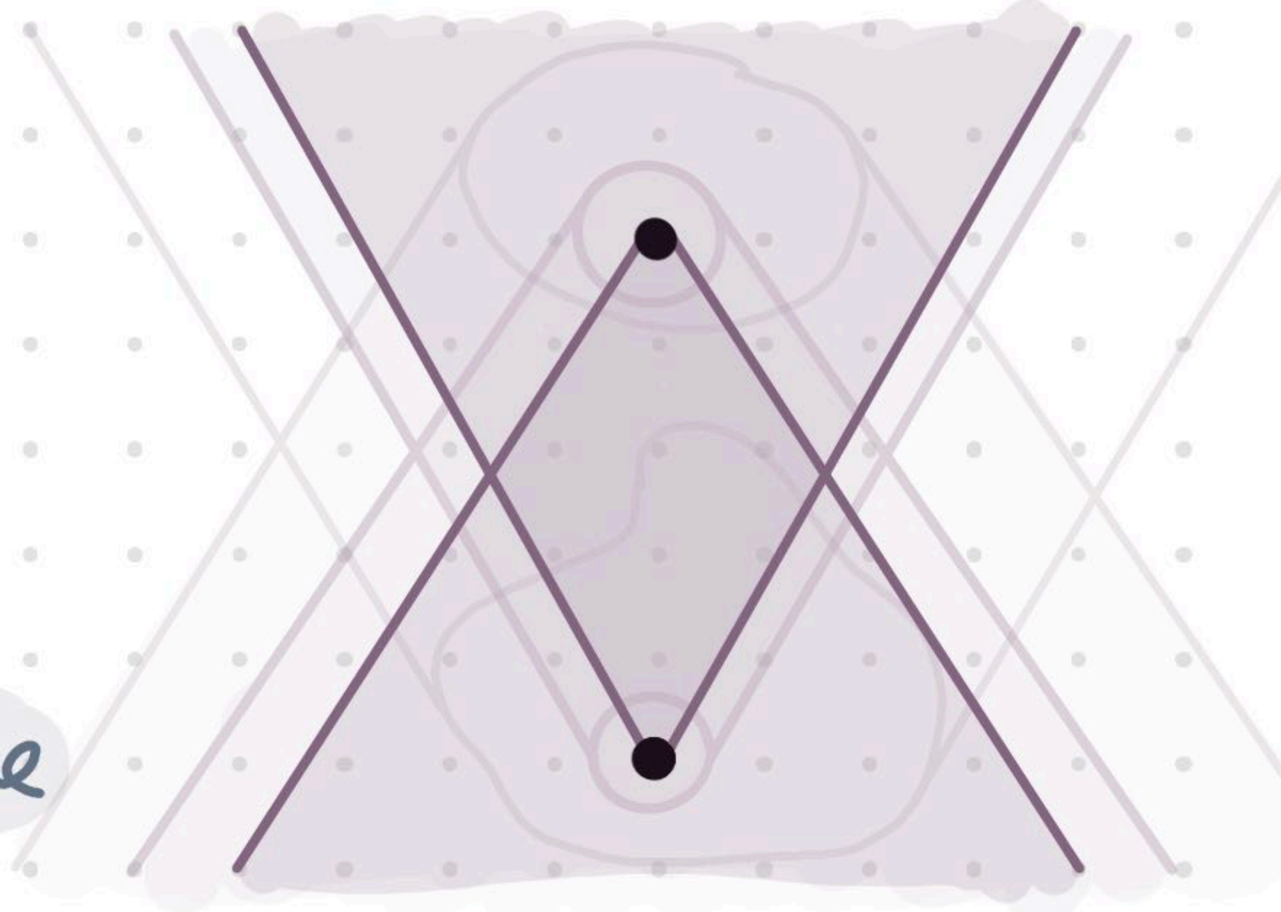


Adjunction

$$
\begin{tikzcd}
\mathbf{OrdTop}^{\bullet}_{oc} \arrow[r, shift left=2, "Loc"] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc}^{\bullet} \arrow[l, shift left=2, "pt"] \\
(S, \leq) \arrow[r, mapsto] & (Loc(S), \trianglelefteq) \\
(pt(X), \leq) & (X, \trianglelefteq) \arrow[l, mapsto]
\end{tikzcd}
$$

THEOREM

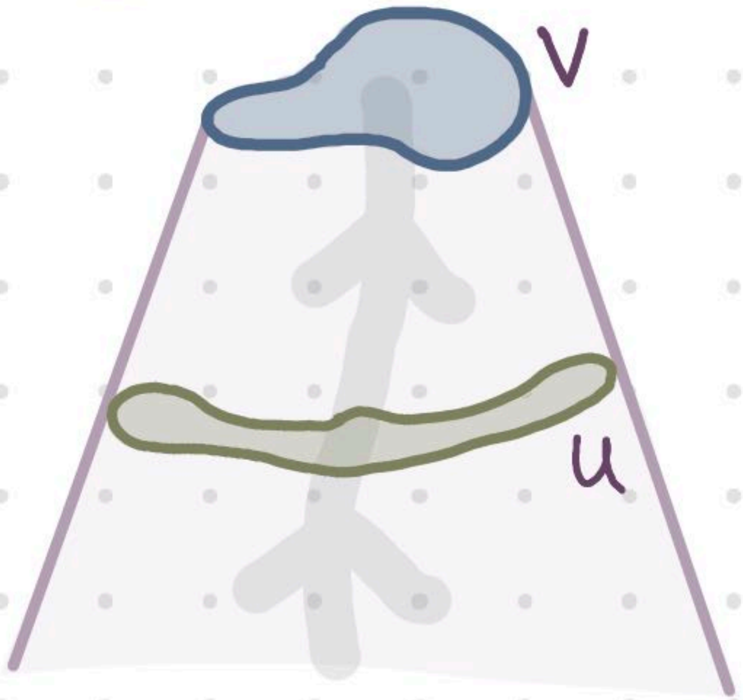
$$\mathbf{OrdTop}^{\bullet}_{OC} \underset{\mathrm{pt}}{\overset{\mathrm{Loc}}{\rightleftarrows}} \mathbf{OrdLoc}^{\bullet} \qquad (\perp)$$

COROLLARY

$$\left\{\begin{array}{c}\text{sober } T_0\text{-ordered}\\ \text{spaces with OC}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{spatial ordered}\\ \text{locales with } \bullet\end{array}\right\}$$

Causal Coverage

[Christensen, Crane 05]

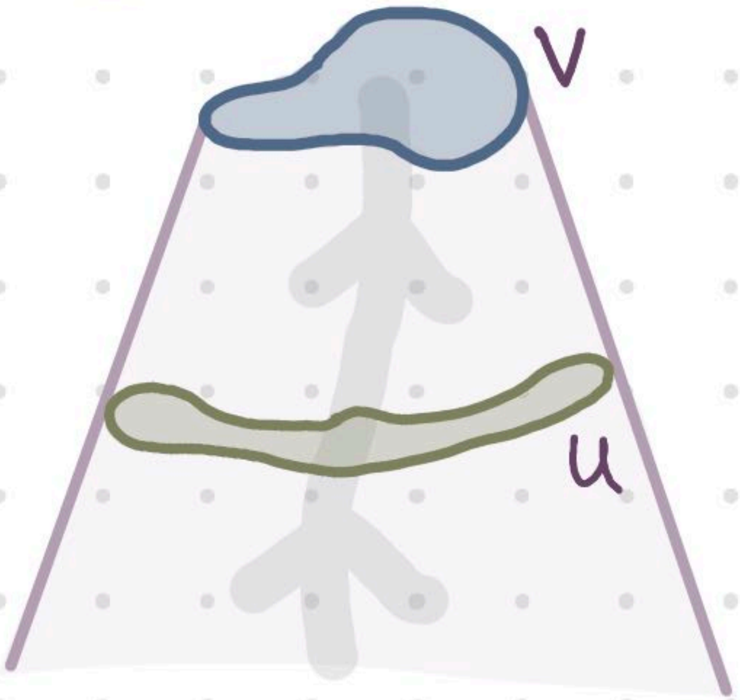


IDEA all info. reaching V must pass through U:

$$U \in \mathrm{Cov}^{-}(V)$$

Causal Coverage

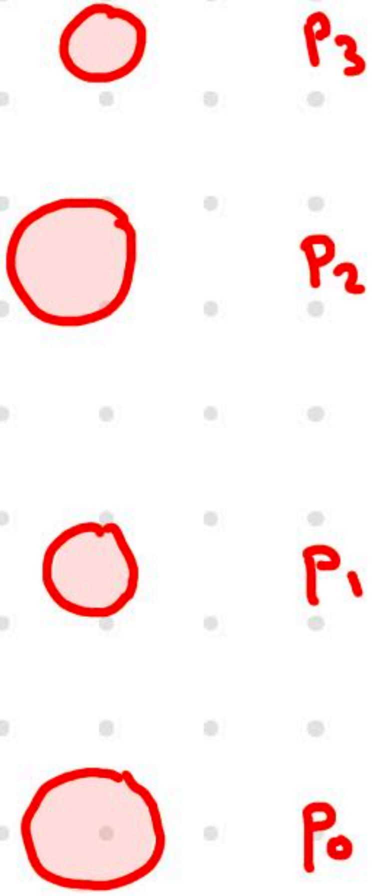
[Christensen, Crane 05]



PATHS

a path is a finite chain:

$$p_0 \trianglelefteq p_1 \trianglelefteq \cdots \trianglelefteq p_{N-1} \trianglelefteq p_N$$



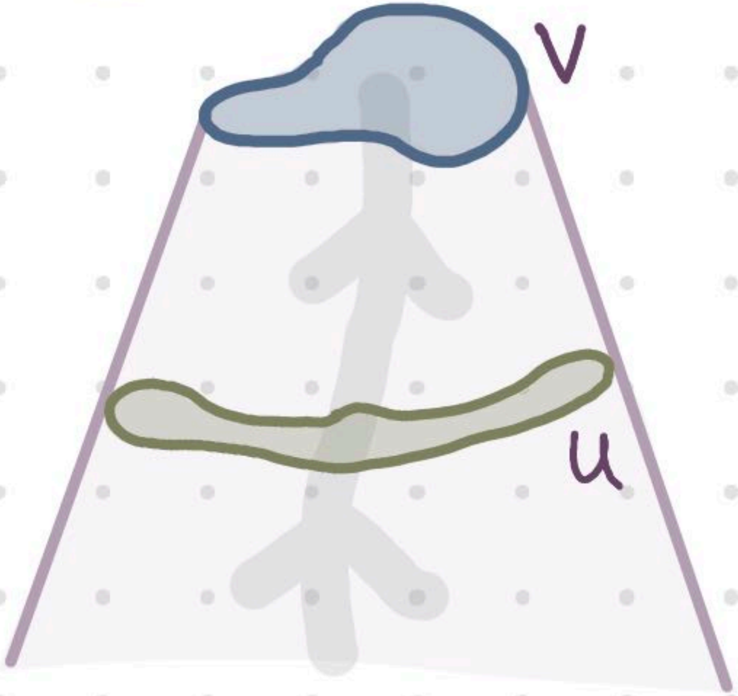
IDEA

all info. reaching V **must** pass through U:

$$U \in \mathrm{Cov}^-(V)$$

Causal Coverage

[Christensen, Crane 05]



PATHS

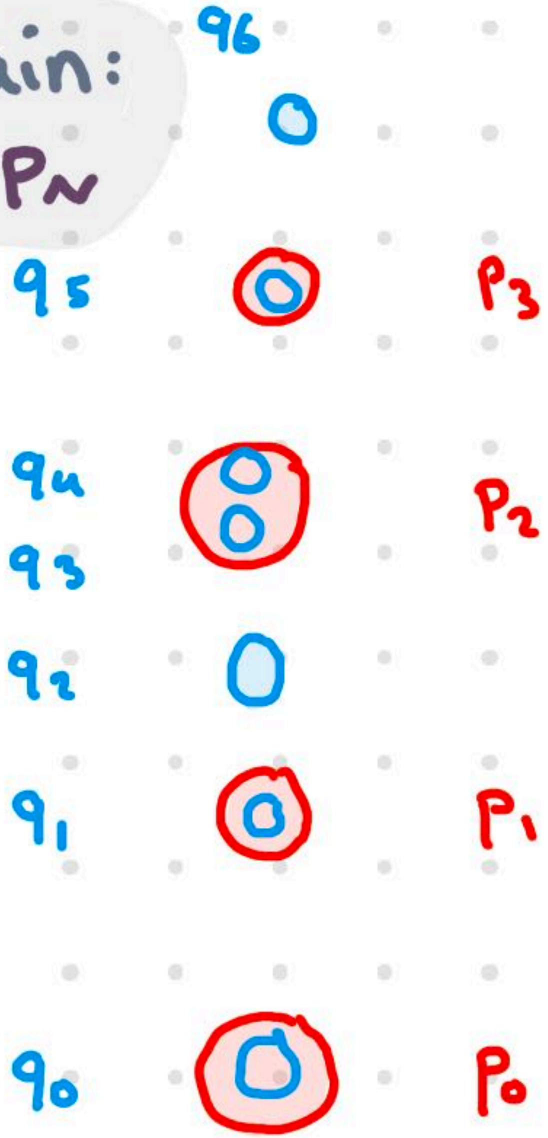
a path is a finite chain:

$$p_0 \trianglelefteq p_1 \trianglelefteq \cdots \trianglelefteq p_{N-1} \trianglelefteq p_N$$

REFINES

q refines p if:

$$\forall n \, \exists m : q_m \sqsubseteq p_n .$$



IDEA

all info. reaching V **must** pass through U:

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Causal Coverage

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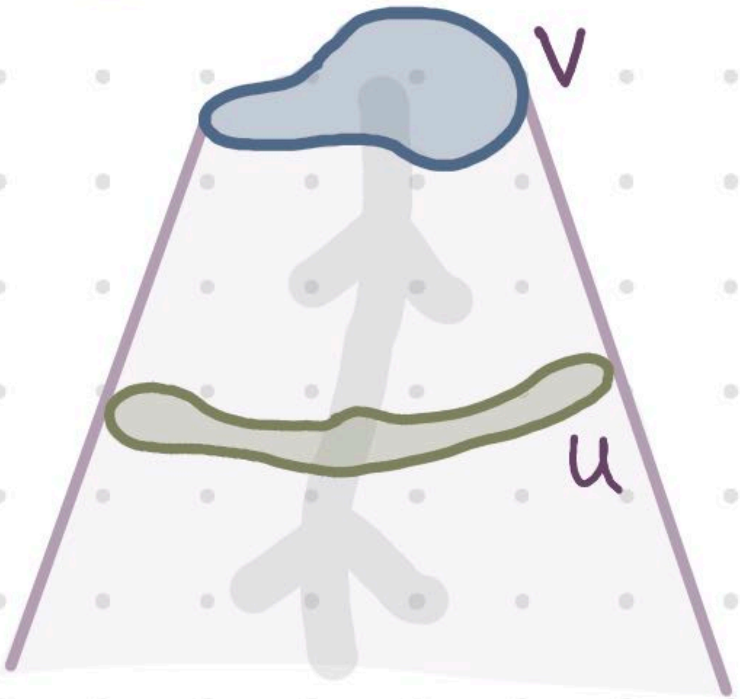
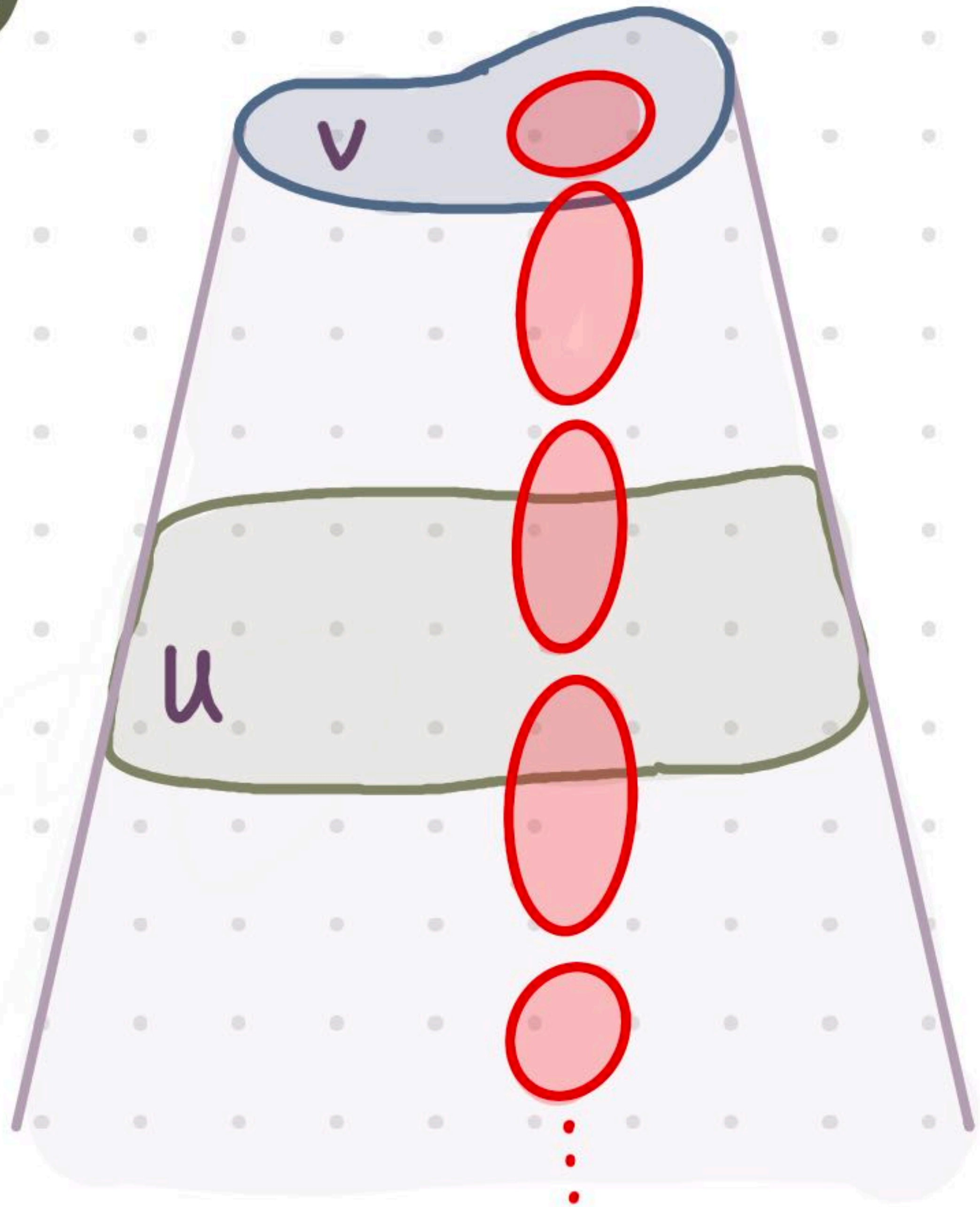


IDEA

all info. reaching V must pass through U:

$$U \in \mathrm{Cov}^-(V)$$



Causal Coverage

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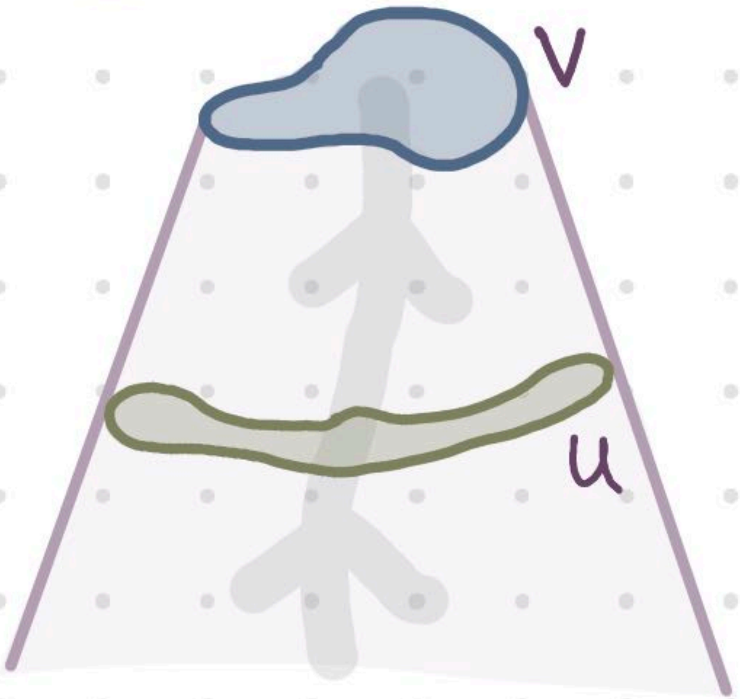

IDEA: all info. reaching V must pass through U:

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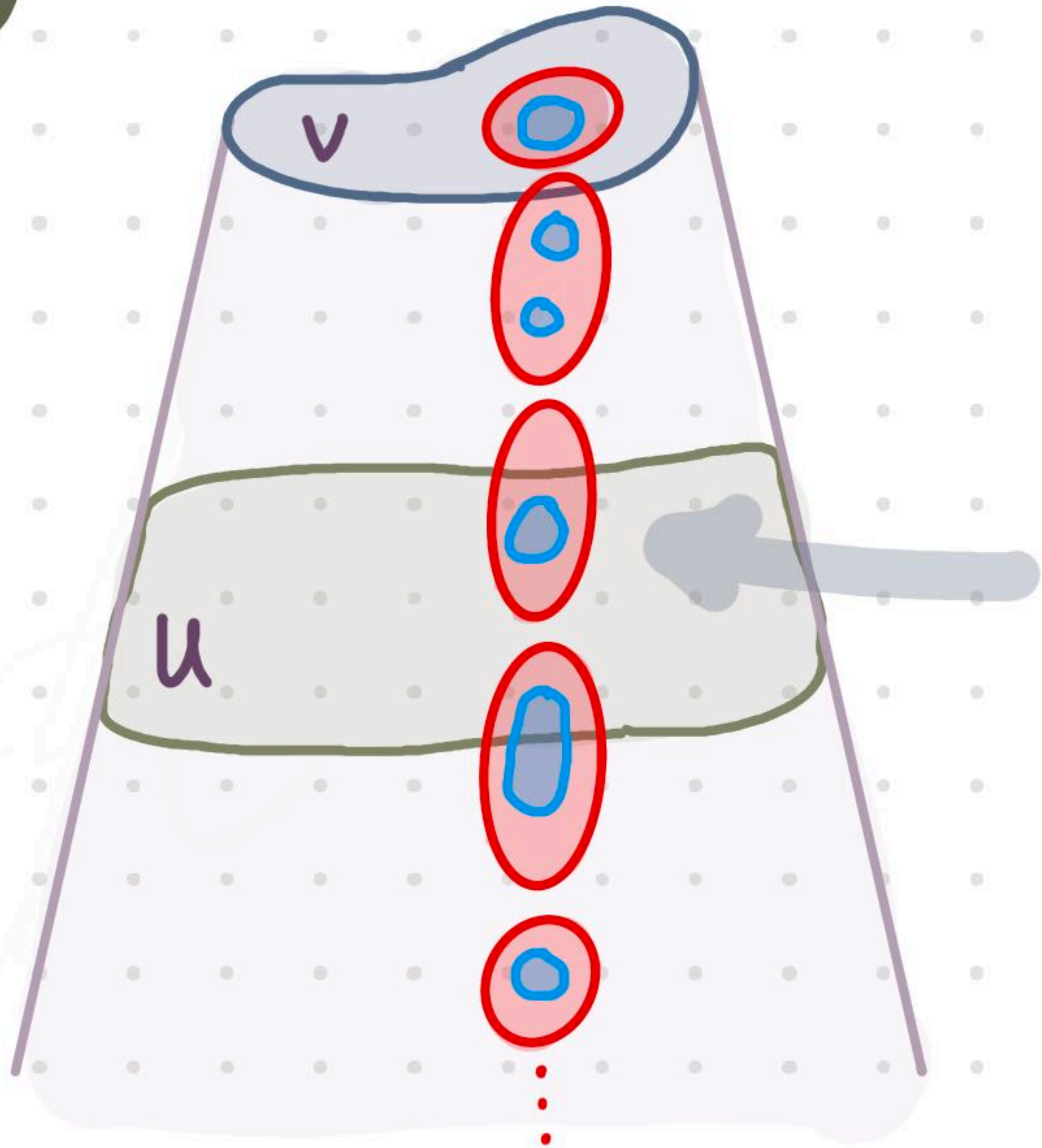

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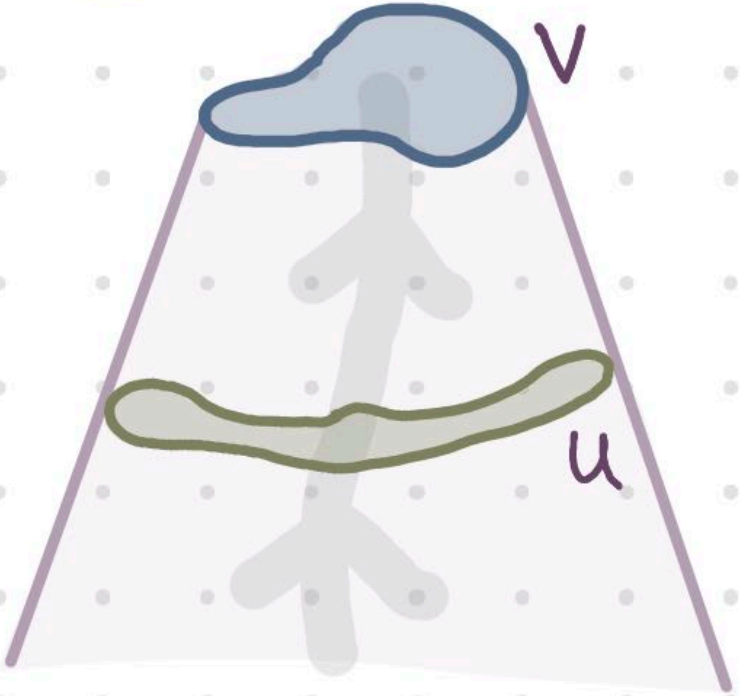
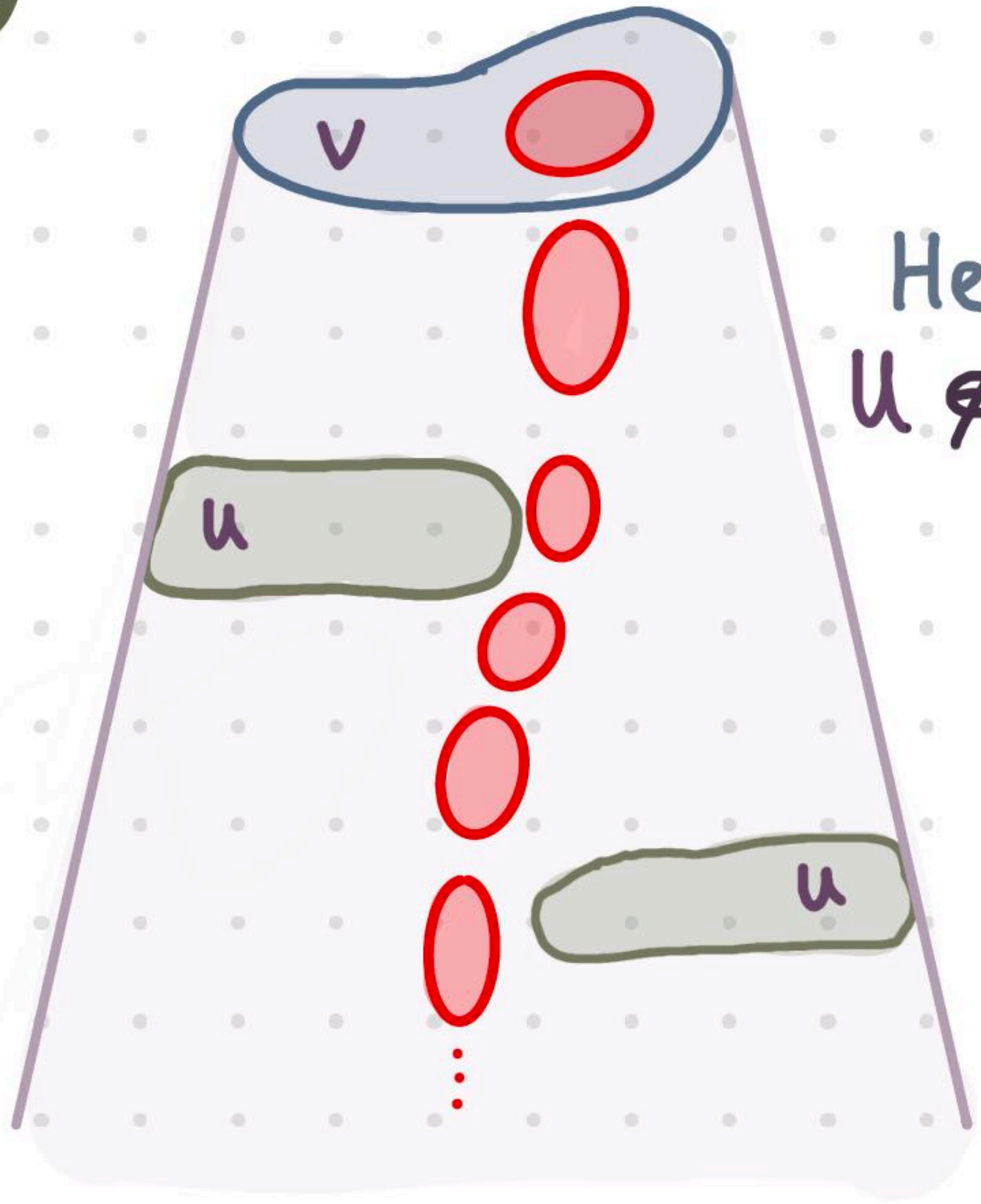


IDEA

all info. reaching V must pass through U:

$U \in Cov^{-}(V)$

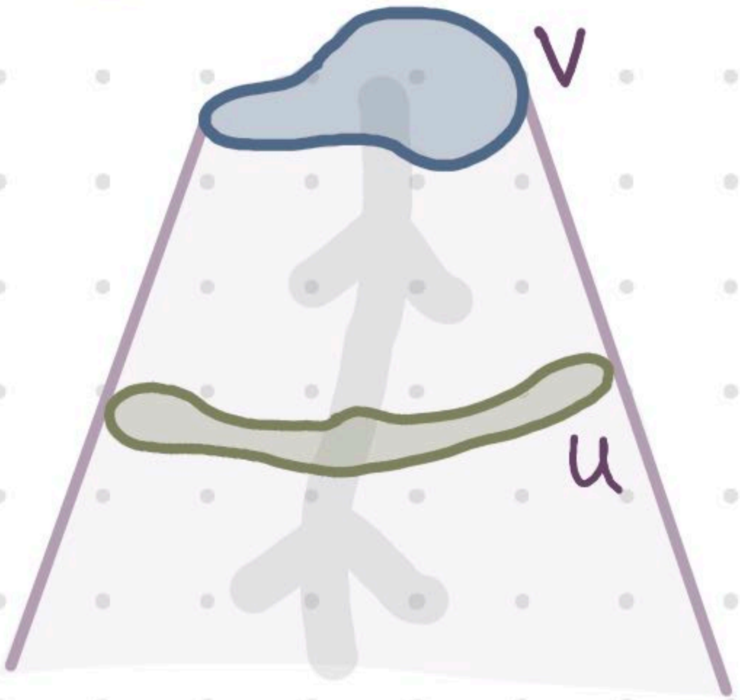


Here:

$U \notin Cov^{-}(V)$

Causal Coverage

[Christensen, Crane 05]



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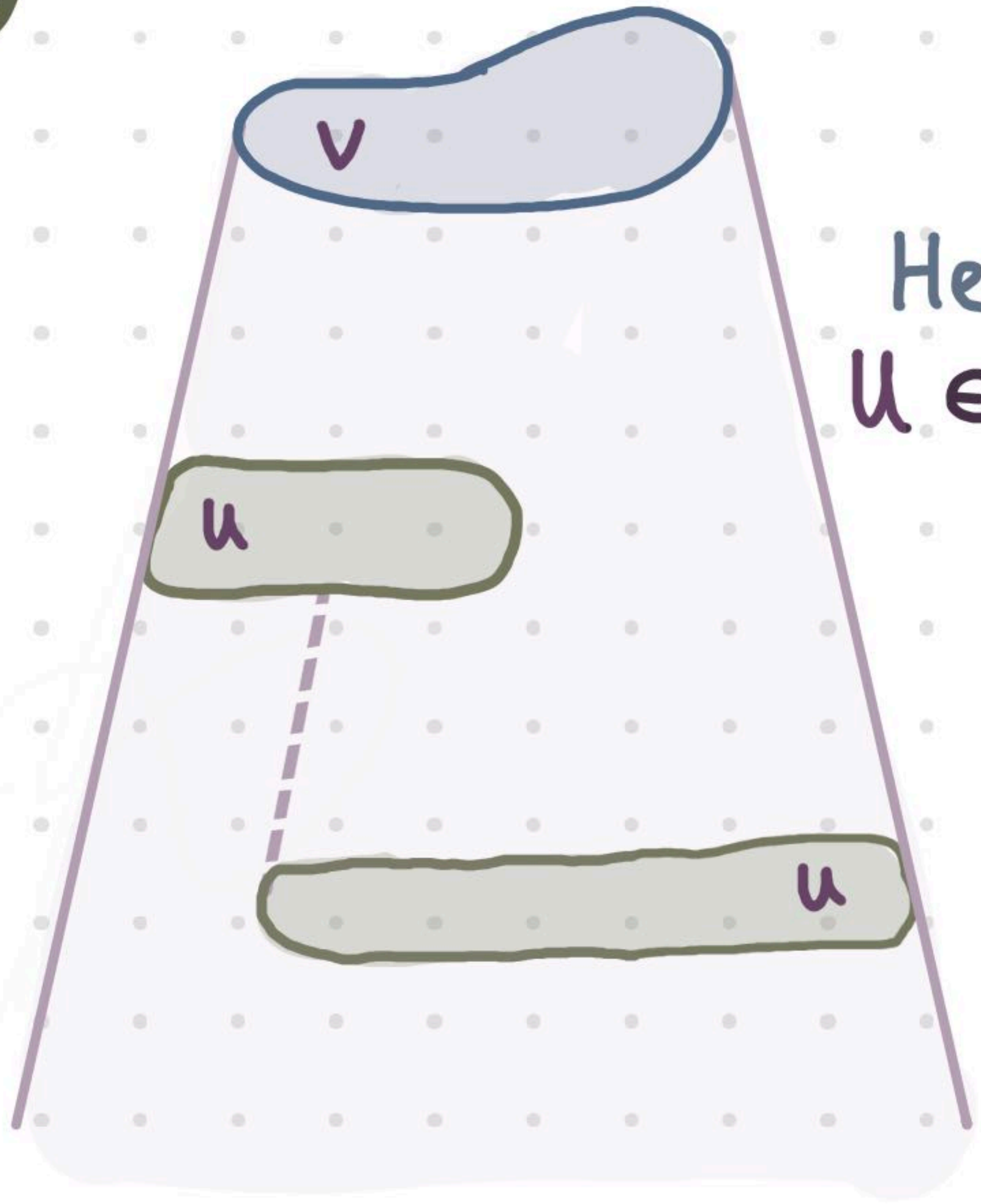
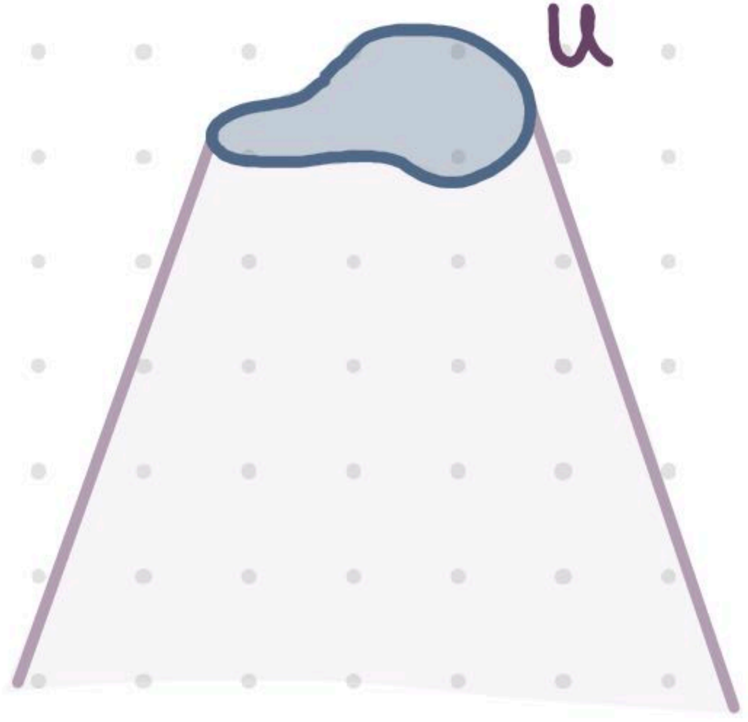


Here: $U \in \mathrm{Cov}^-(V)$

Causal Coverage

LEMMA

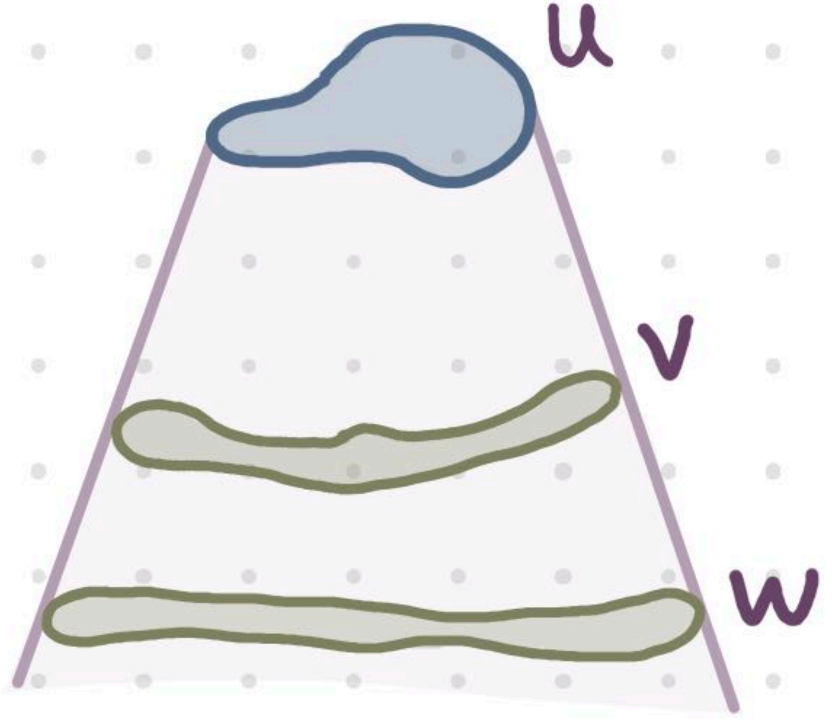
- $U \in Cov^{-}(U)$
- $\downarrow U \in Cov^{-}(U)$



Causal Coverage

LEMMA

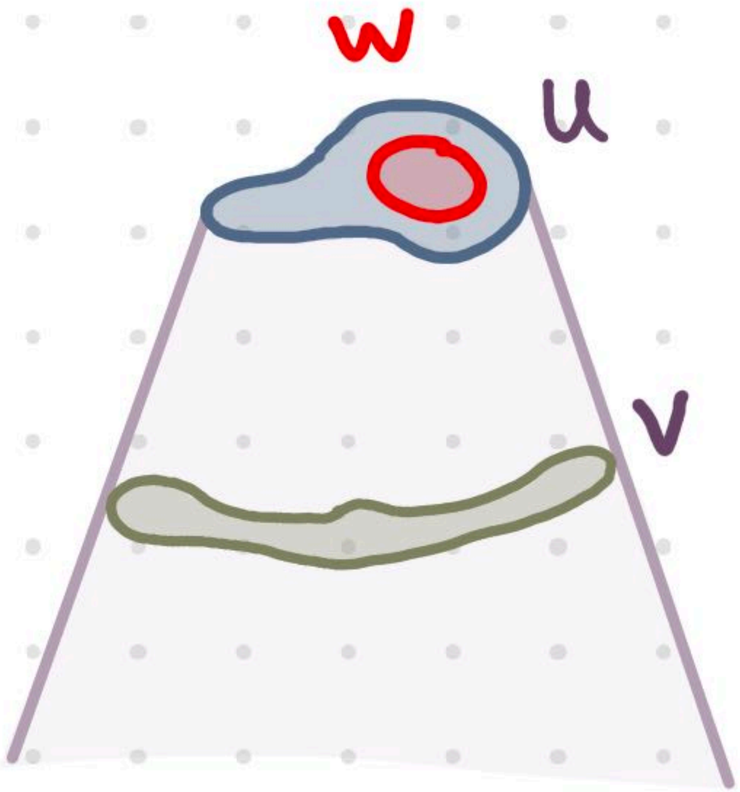
- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V),\ V \in \mathrm{Cov}^-(U)$
  $\Longrightarrow W \in \mathrm{Cov}^-(U)$



Causal Coverage

LEMMA

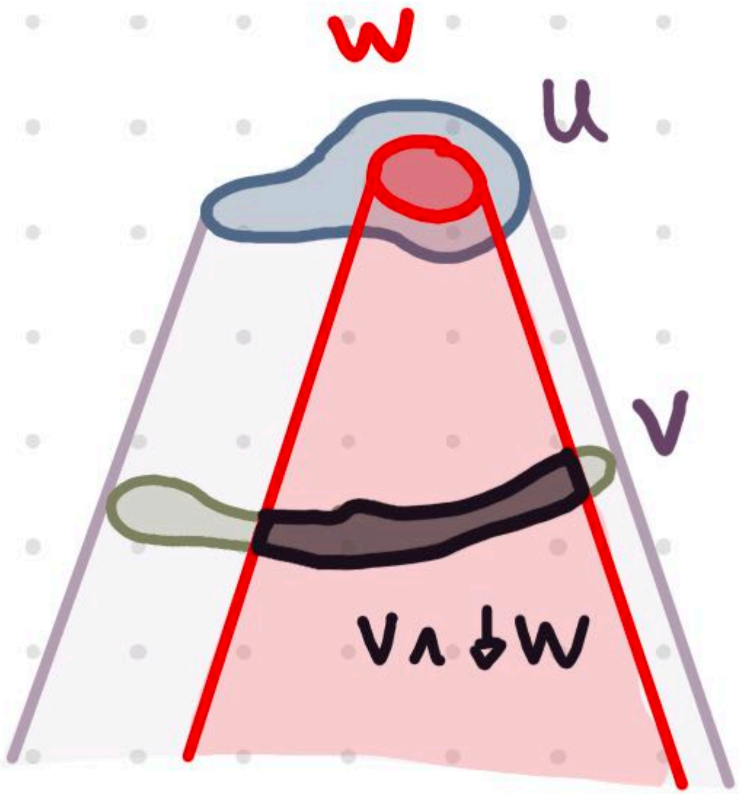
- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V),\ V \in \mathrm{Cov}^-(U)$
  $\Longrightarrow\ W \in \mathrm{Cov}^-(U)$
- $V \in \mathrm{Cov}^-(U),\ W \subseteq U$
  $\Longrightarrow\ V \wedge \downarrow W \in \mathrm{Cov}^-(W)$



Causal Coverage

LEMMA

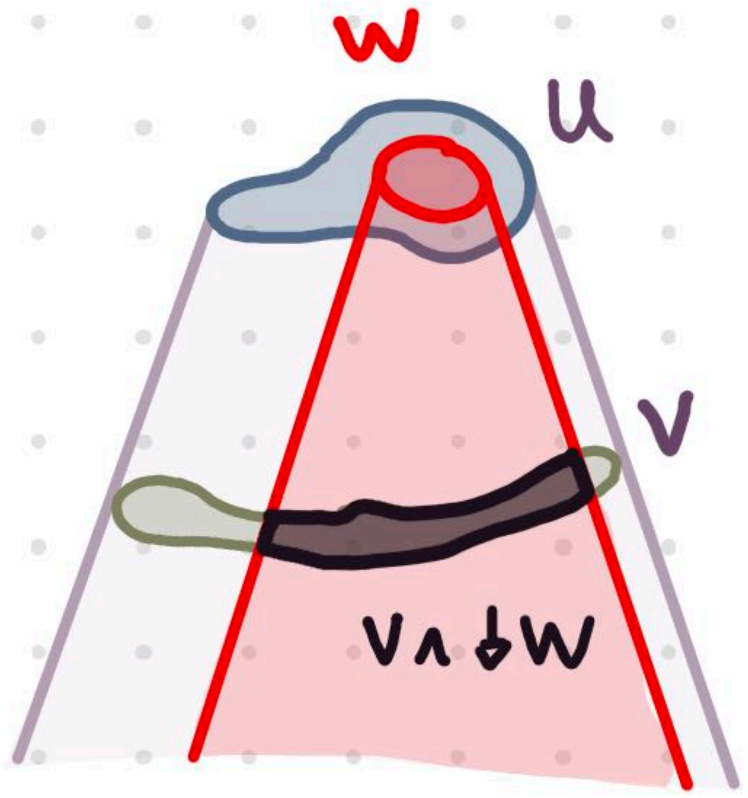
- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V),\ V \in \mathrm{Cov}^-(U)$
  $\implies W \in \mathrm{Cov}^-(U)$
- $V \in \mathrm{Cov}^-(U),\ W \sqsubseteq U$
  $\implies V \wedge \downarrow W \in \mathrm{Cov}^-(W)$



Causal Coverage

LEMMA

- $U \in \mathrm{Cov}^-(U)$
- $\downarrow U \in \mathrm{Cov}^-(U)$
- $W \in \mathrm{Cov}^-(V)$, $V \in \mathrm{Cov}^-(U)$
  $\Longrightarrow W \in \mathrm{Cov}^-(U)$
- $V \in \mathrm{Cov}^-(U)$, $W \sqsubseteq U$
  $\Longrightarrow V \wedge \downarrow W \in \mathrm{Cov}^-(W)$



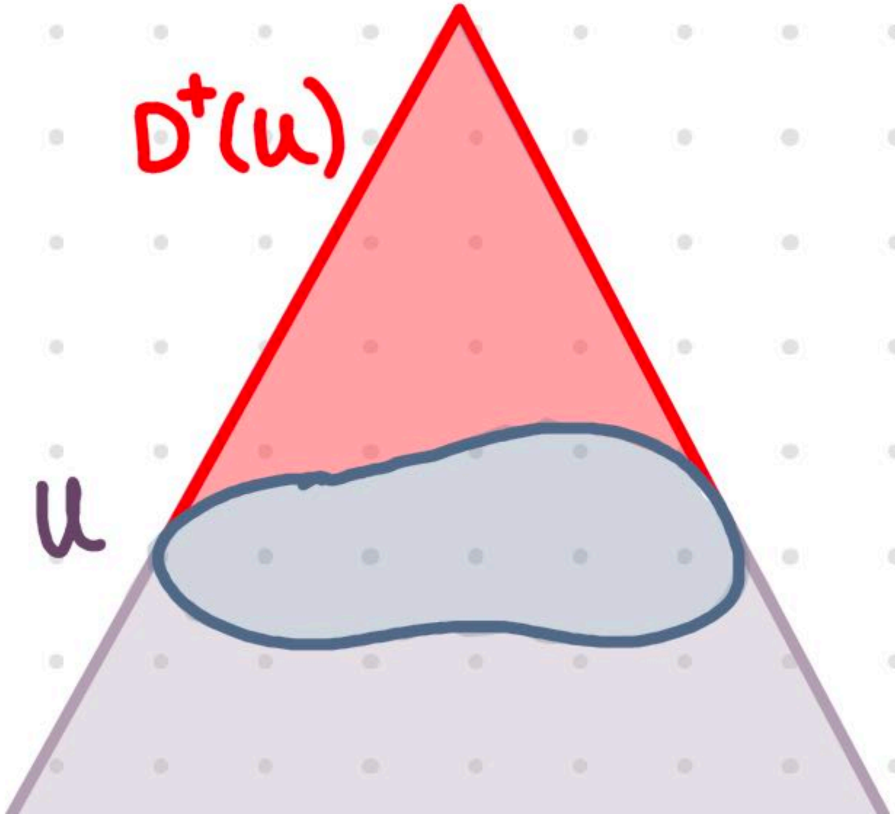
Grothendieck topology on $\mathcal{K}\ell(\downarrow)$

Causal Coverage

[Geroch 70]

DOMAIN OF DEPENDENCE

$$D^+(u) := \bigvee \{ w \in \mathcal{O}X : u \in \mathrm{Cov}^-(w) \}$$



EXAMPLE

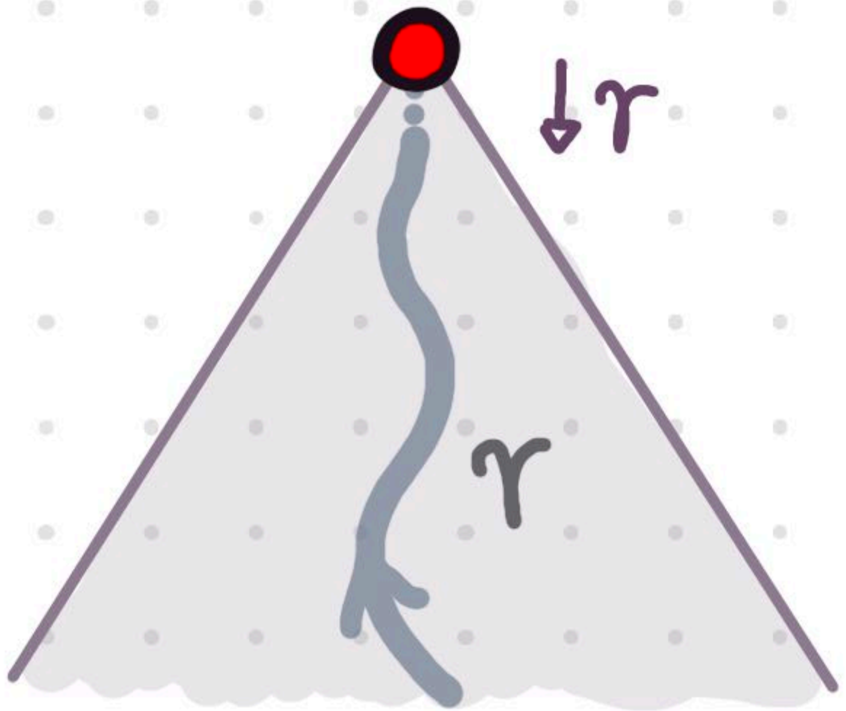
Sheaf of solutions to the wave equation on Minkowski space

Q. $\mathrm{Sh}(X, \trianglelefteq)$? Internal logic related to modal logic approaches of spacetime ? [Goldblatt 80, 92]

Causal Boundaries

joint with:
Prakash Panangaden

IDEA: add ideal points to space(time) via would-be limits of curves γ



Causal Boundaries

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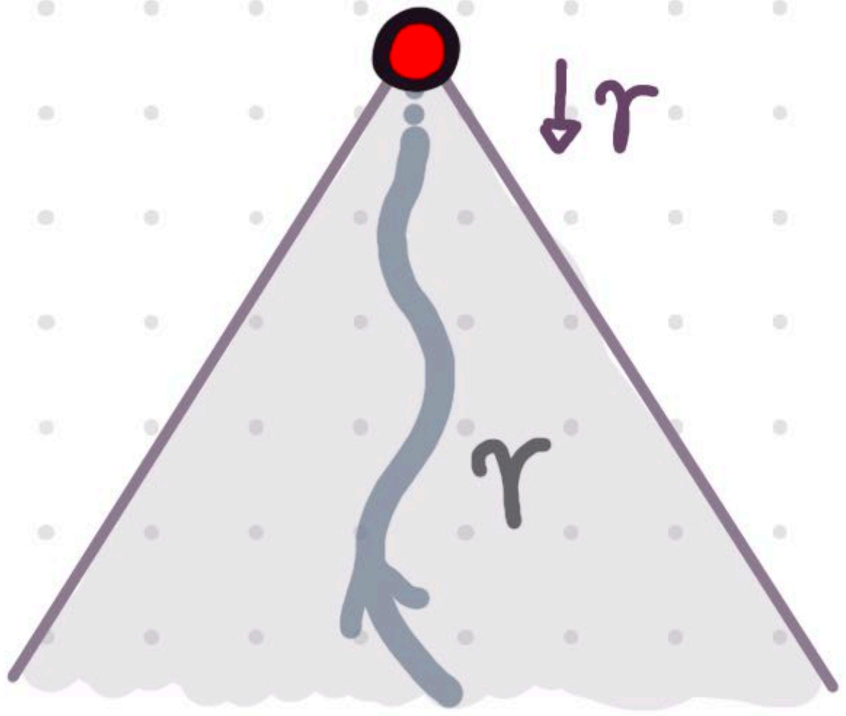
IDEA: add ideal points to space(time) via would-be limits of curves γ

DEFN. $\mathrm{IP}(M) := \{ \downarrow\gamma : \gamma \text{ timelike curve} \}$



Causal Boundaries

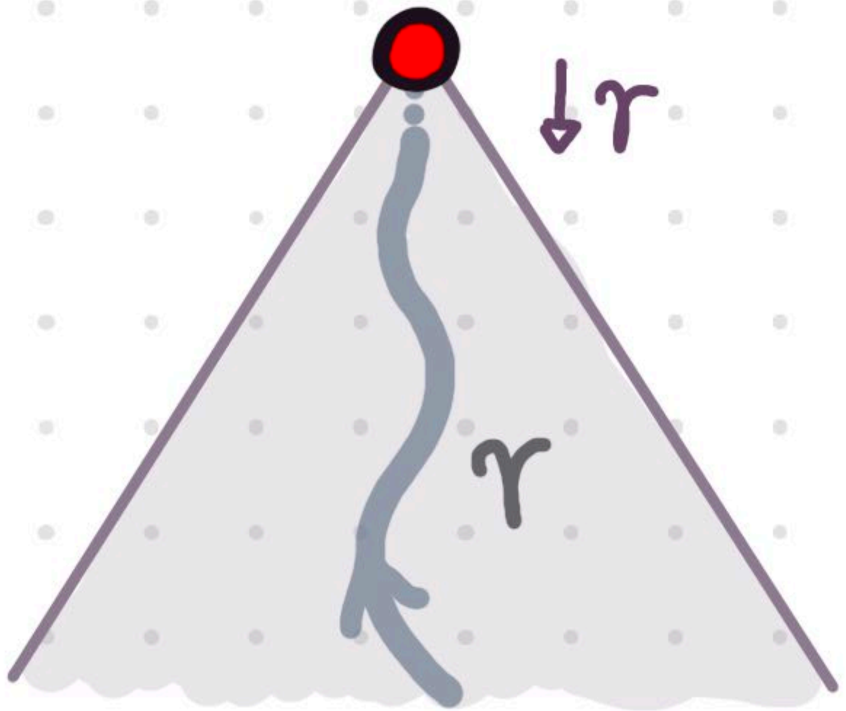
joint with: Prakash Panangaden

IDEA add ideal points to space(time) via would-be limits of curves γ

DEFN. $IP(M) := \{ \downarrow\gamma : \gamma \text{ timelike curve} \}$

$I^-(x) \in IP(M) \rightsquigarrow x \in M$

$M \in IP(M) \rightsquigarrow$ future timelike infinity



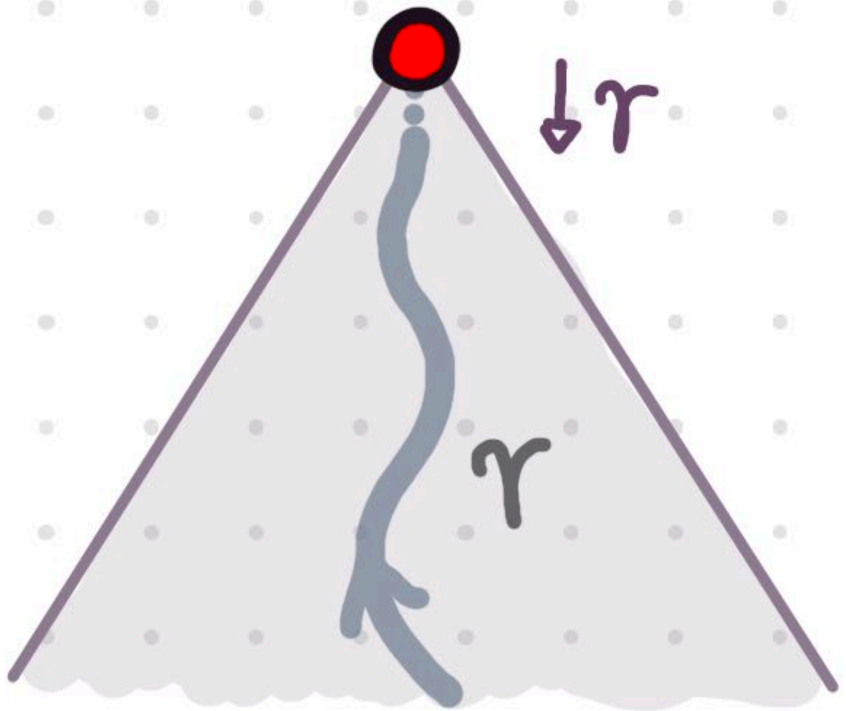
Causal Boundaries

joint with:
Prakash Panangaden

DEFN.

a coprime is $\emptyset \neq P = \downarrow P$ s.t.

$\downarrow A \vee \downarrow B = P \implies P = \downarrow A$ or $\downarrow B$



THEOREM

$P \in \mathrm{im}(\downarrow)$ is coprime

iff

$\exists$ causal curve γ s.t. $P = \downarrow\gamma$

[Geroch, Kronheimer, Penrose 72]

Causal Boundaries

$\mathrm{im}(\uparrow)$ is a frame,
so defines locale M^Δ.

LEMMA

there is a bijection

$$\mathrm{IP}(M) \cong \mathrm{pt}(M^\Delta)$$

$$P \longmapsto \neg P$$

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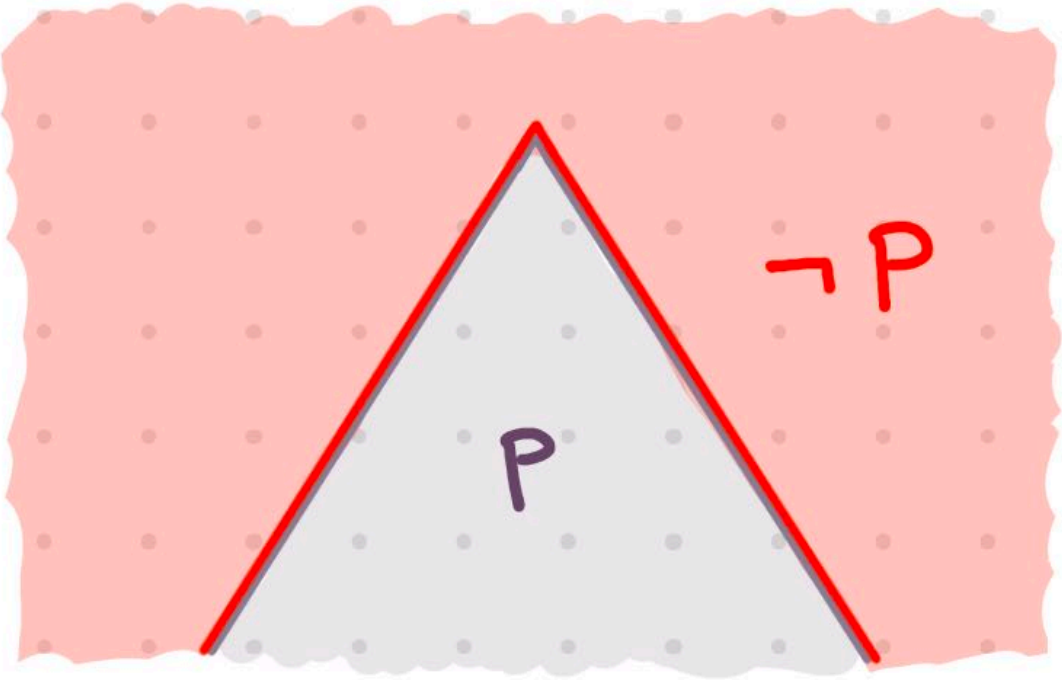

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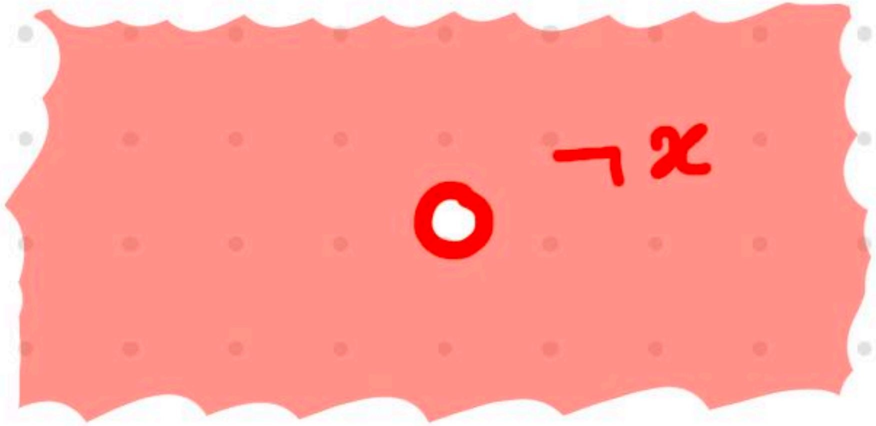


Causal Boundaries

E.G. $c=0$ Minkowski : N

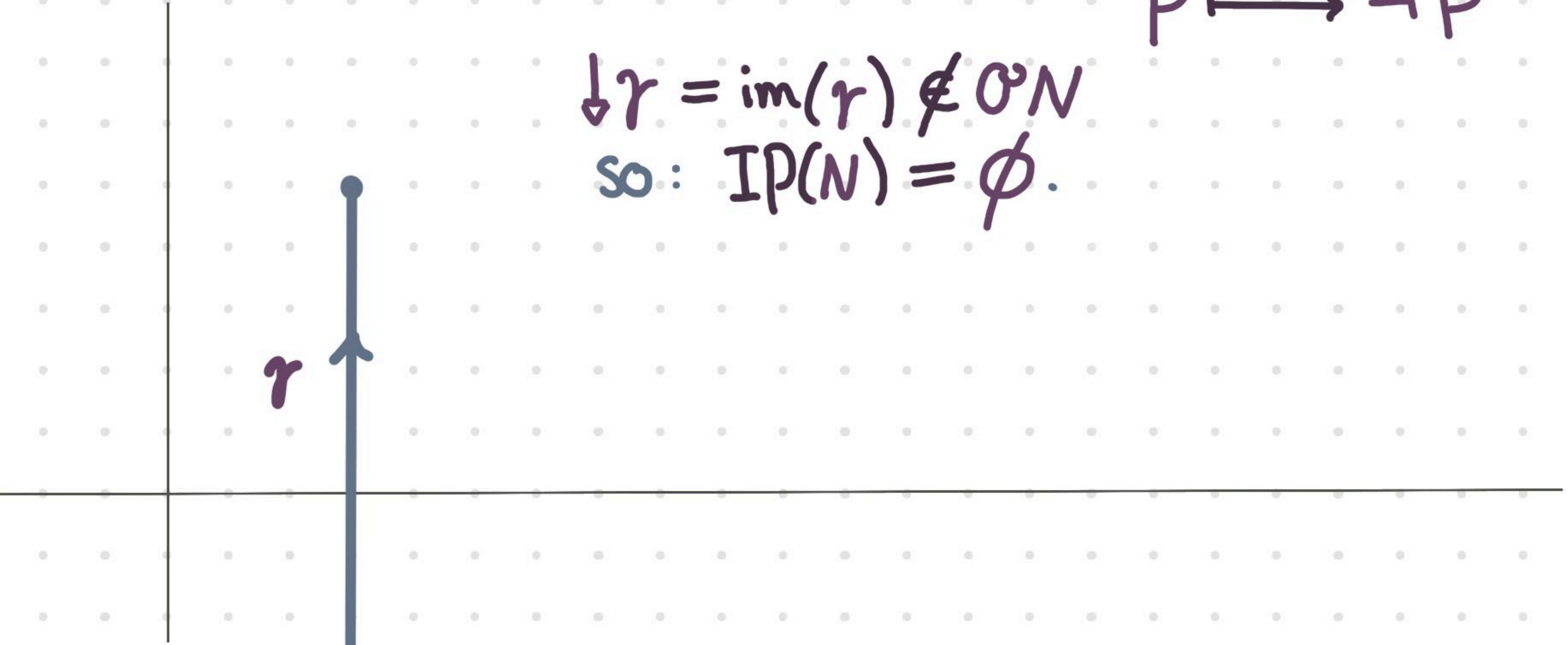
LEMMA there is a bijection

$$IP(M) \cong pt(M^{\Delta})$$

$$P \longmapsto \neg P$$

$$\downarrow\gamma = \mathrm{im}(\gamma) \notin O'N$$

so: $IP(N) = \emptyset$.



Causal Boundaries

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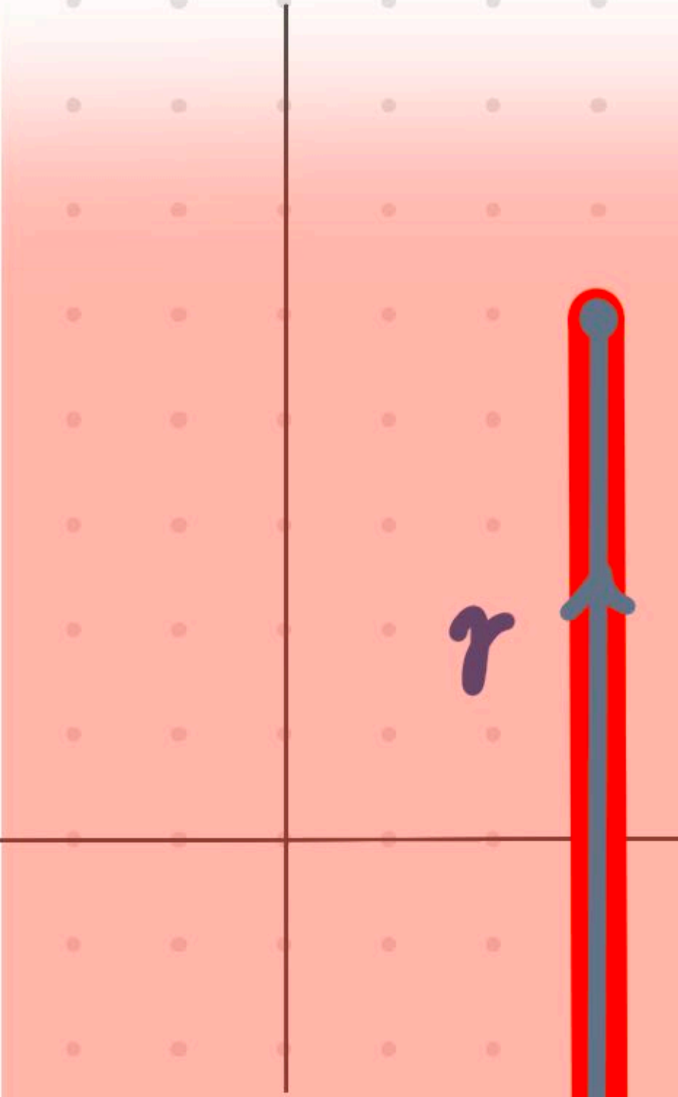


$$\downarrow\gamma = \mathrm{im}(\gamma) \notin \mathcal{O}^{\circ}N$$

so: $IP(N) = \emptyset$.

$$\neg\downarrow\gamma \in pt(N^{\Delta}) \cong \mathbb{R}^2 \sqcup \mathbb{R}$$

Causal Boundaries

LEMMA There is an isomorphism:

$$\left\{\begin{array}{c}\text{"convex" ordered}\\ \text{locales s.t. } \uparrow, \downarrow\\ \text{ioin-preserving}\end{array}\right\} \cong \left\{\begin{array}{c}\text{"reflective"}\\ \text{biframes}\end{array}\right\}^{op}$$

$$(X, \trianglelefteq) \longmapsto \big(\mathcal{O}X, \mathrm{im}(\uparrow), \mathrm{im}(\downarrow)\big)$$

Q. biframe compactifications for ordered locales?

Signalling?

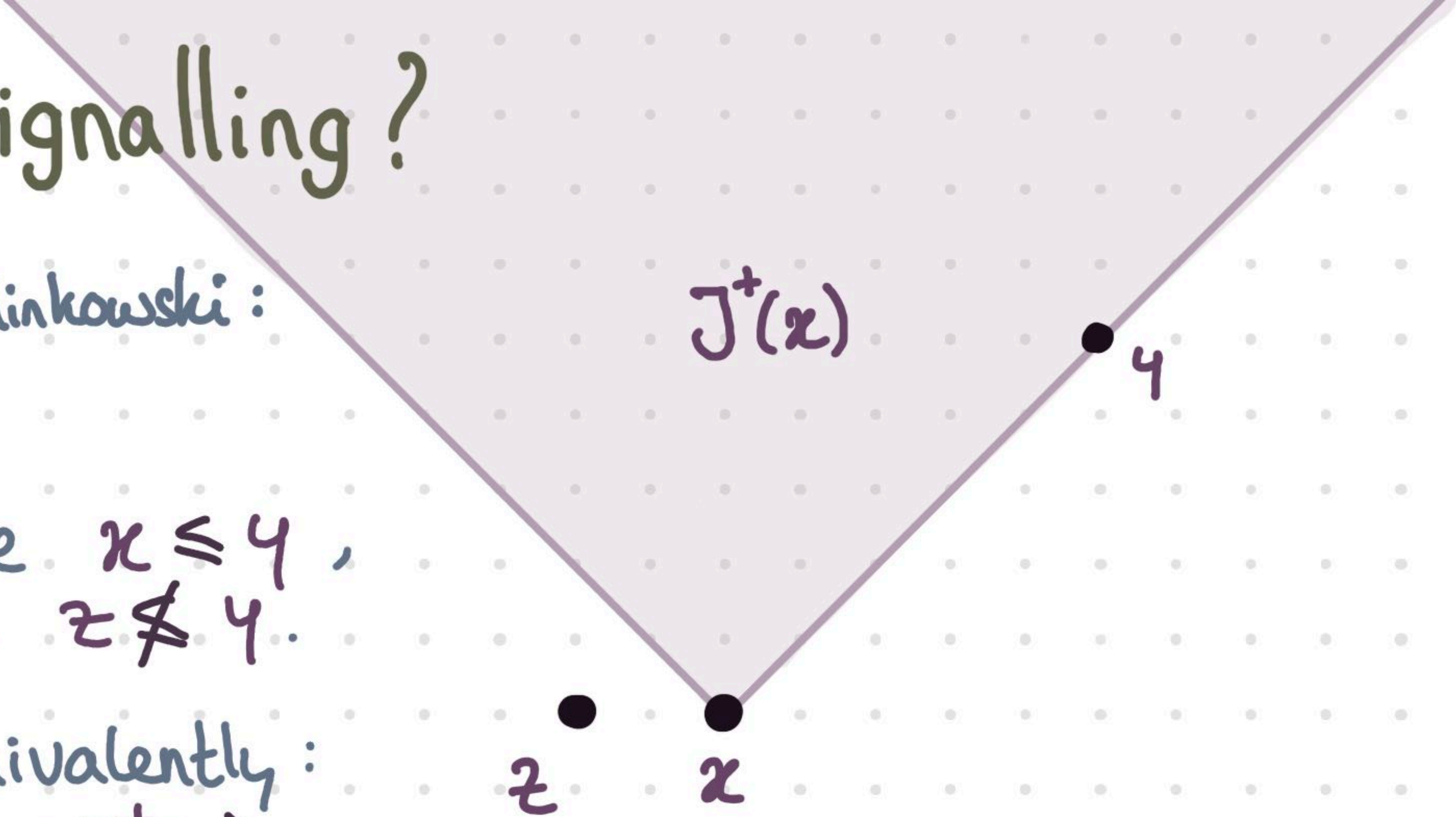
spaces S	locales X
open subset $U \in \mathcal{O}S$	open region $U \in \mathcal{O}X$
arbitrary subset $A \in \mathcal{P}(S)$	arbitrary sublocale $A \in \mathcal{S}\ell(X)$

E.g. take $\mathbb{Q}, \mathbb{I} \subseteq \mathbb{R}$.
then $\mathbb{Q} \cap \mathbb{I} = \emptyset$,
but: $\tilde{\mathbb{Q}} \wedge \tilde{\mathbb{I}} \neq \emptyset$.

Subset $A \subseteq S$ induces sublocale $\tilde{A} \in \mathcal{S}\ell(\mathrm{Loc}(S))$.

Signalling?

In Minkowski:



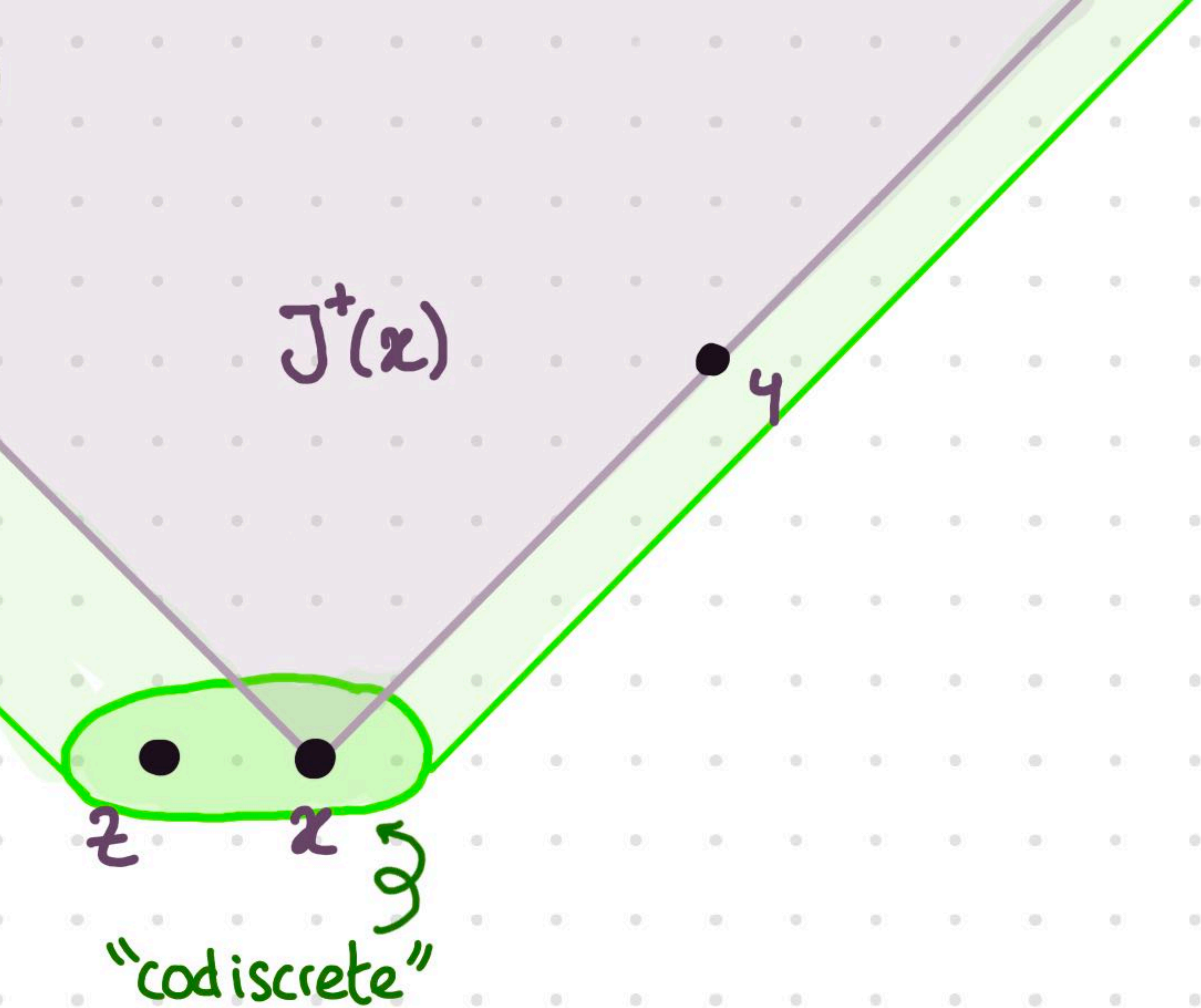
Here $x \leq y$,
but $z \not\leq y$.

Equivalently:
$y \in J^+(x)$,

$y \notin J^+(z)$.

Signalling?

In Minkowski:



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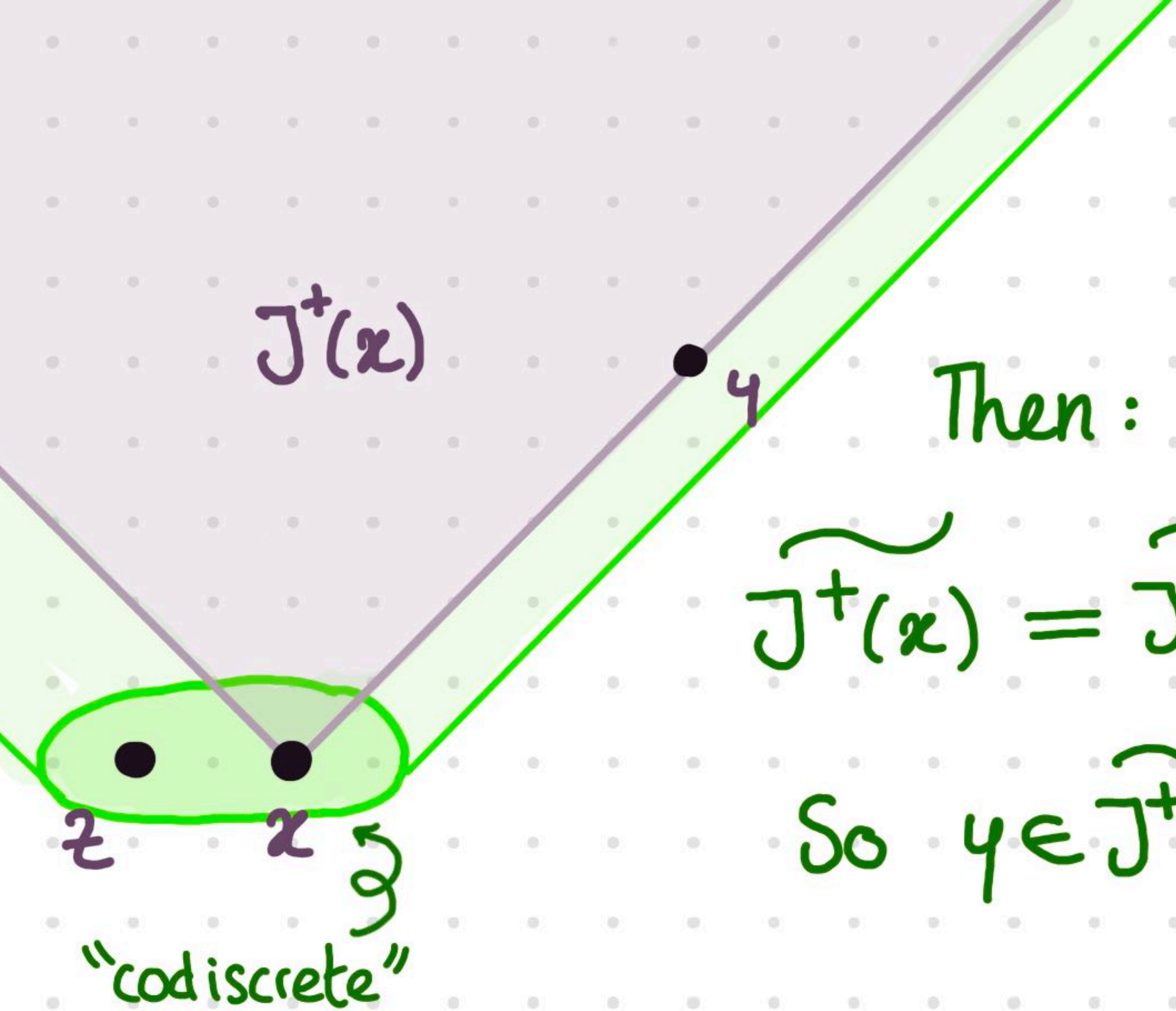


Then:

$\widetilde{J^+(x)} = \widetilde{J^+(z)}$,

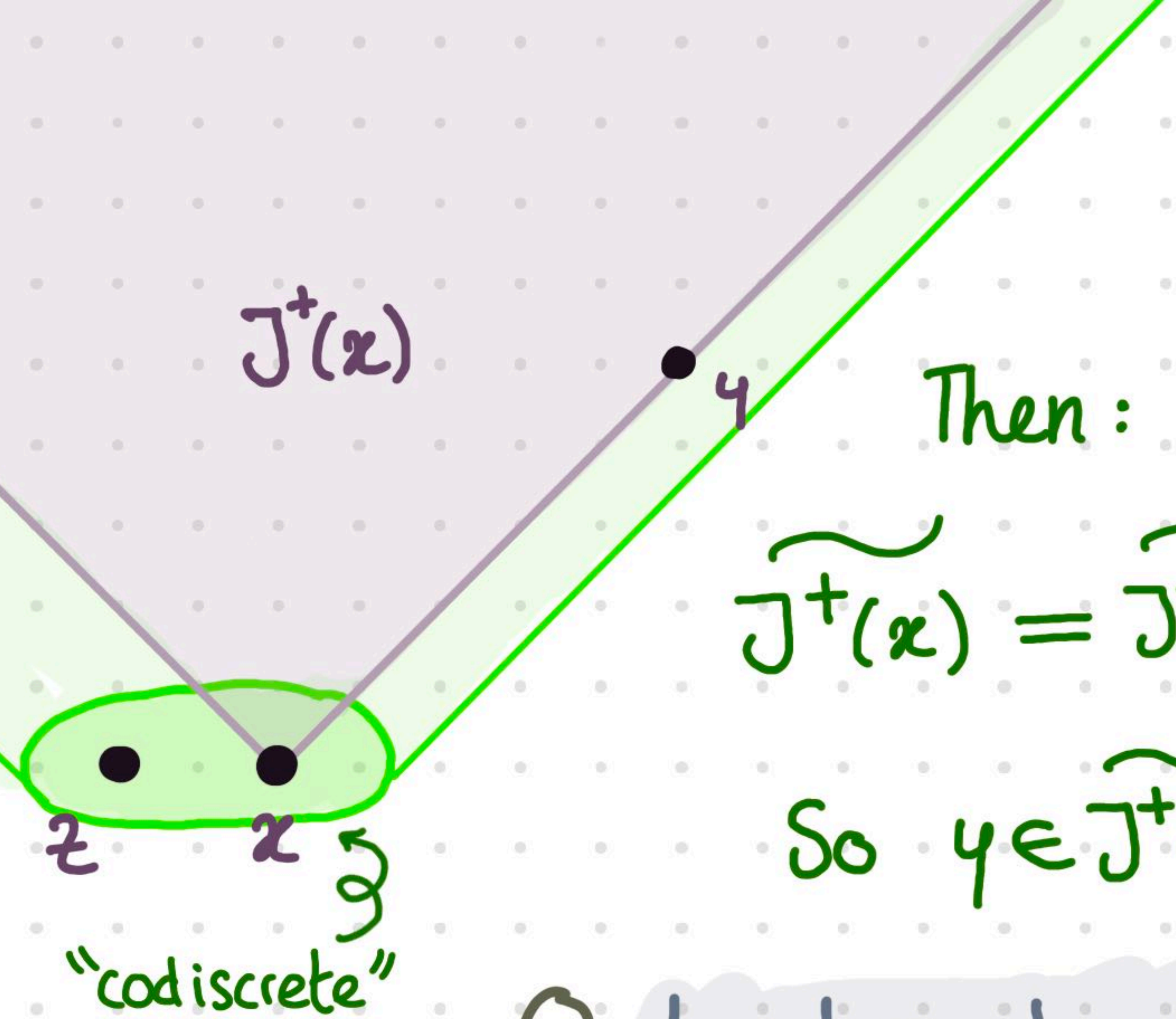
So $y \in \widetilde{J^+(z)}$.

Signalling?

In Minkowski:

Here $x \leq y$, but $z \not\leq y$.

Equivalently: $y \in J^+(x)$, $y \notin J^+(z)$.



Then:

$\widetilde{J^+(x)} = \widetilde{J^+(z)}$,

So $y \in \widetilde{J^+(z)}$.

Q. topology change from entanglement?

AQFTs

HAAG-KASTLER

a functor

$$\mathcal{A} : \mathcal{O}M \longrightarrow \mathrm{Alg} \; ; \qquad U \longmapsto \mathcal{A}(U).$$

such that:

if $U, V \in \mathcal{O}M$ spacelike separated, then

$$[\mathcal{A}(U), \mathcal{A}(V)] = 0 \text{ in } \mathcal{A}(M).$$

AQFTs

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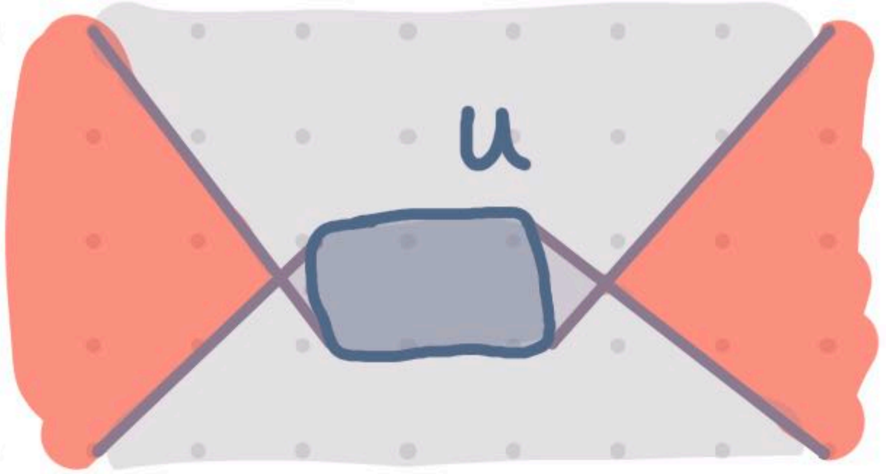
CAUSAL COMPLMT.

for $U \in \mathcal{O}X$ define:

$$U^{\perp} := \neg(\uparrow U \vee \downarrow U)$$
$$= \neg \uparrow U \wedge \neg \downarrow U.$$



$U^{\perp}$

AQFTs

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CAUSAL COMPLMT.

for $U \in \mathcal{O}X$ define:

$$U^\perp := \neg(\uparrow U \vee \downarrow U) = \neg\uparrow U \wedge \neg\downarrow U.$$

$$U \perp V \iff U \sqsubseteq V^\perp \iff V \sqsubseteq U^\perp$$

Heyting Implication

CLASSICAL

if $U \in \mathcal{O}X$ then $U \wedge - : \mathcal{O}X \to \mathcal{O}X$ has adjoint:

$$(U \wedge -) \dashv (U \longrightarrow -).$$

$$U \wedge V \sqsubseteq W \quad \text{iff} \quad V \sqsubseteq U \longrightarrow W$$

Modus Ponens: $U \wedge (U \longrightarrow V) \sqsubseteq V.$

Heyting Implication

"Implication via Spacetime"
[Tabatabai 21]

TEMPORAL

if $U \in \mathcal{O}^\triangledown X$ then $U \wedge \downarrow(-) : \mathcal{O}^\triangledown X \to \mathcal{O}^\triangledown X$ has adjoint:

$$(U \wedge \downarrow(-)) \dashv (U \to_\triangledown -).$$

$$U \wedge \downarrow V \sqsubseteq W \quad \text{iff} \quad V \sqsubseteq U \to_\triangledown W$$

Delayed Modus Ponens: $U \wedge \downarrow(U \to_\triangledown V) \sqsubseteq V.$

Q. Further connection to (modal) logic?

Future Work

Improved adjunction? Internal approach
Constructive?
Full causal bdy. construction
More examples of sheaves
Causal ladder / Globally hyperbolic locales
Recover full geometry via valuation?
Ordered toposes?
Quantum? Via duality $C^*Alg \rightleftarrows$ Quantales
Tensor topology $\Longrightarrow$ CQM