AXIOMS FOR THE CATEGORY OF: HILBERT SPACES & LINEAR CONTRACTIONS

arXiv: 2211.02688

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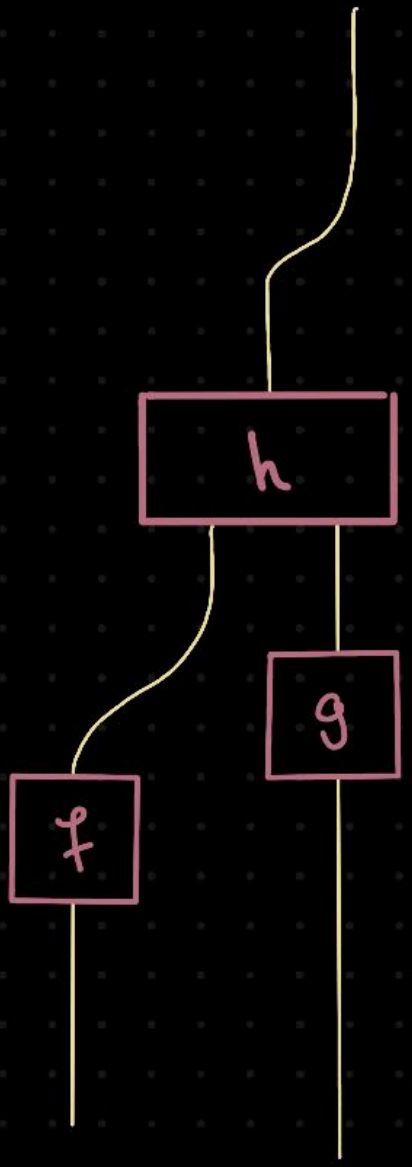
- MOTIVATION (QM)
- THE QUESTION (RECONSTRUCTION)
- RECAP OLD RESULT (HEUNEN + KORNELL)
- PROOF STRATEGY
- SCALARS
- NEW AXIOMS
- MAIN CONSTRUCTION
- THE THEOREM

PHYSICS AS PROCESSES.



WE STUDY PHYSICS THROUGH PROCESSES : $f, g, h \ldots$

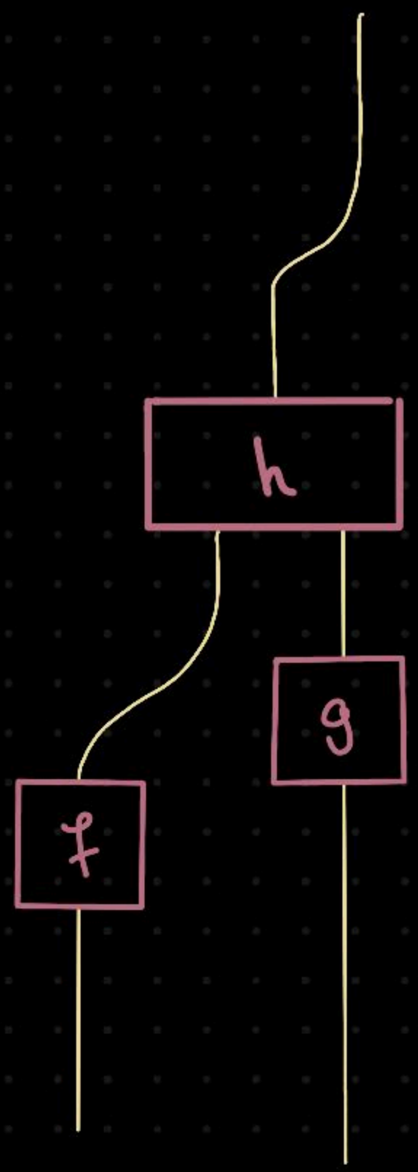
PROCESSES HAPPEN BETWEEN SYSTEMS: $H, K \ldots$

TWO TYPES OF COMPOSITION:

SEQUENTIAL: $g \circ f$ ⇝ CATEGORIES

PARALLEL: $g \otimes f$ ⇝ MONOIDAL CATEGORIES

PHYSICS AS PROCESSES.

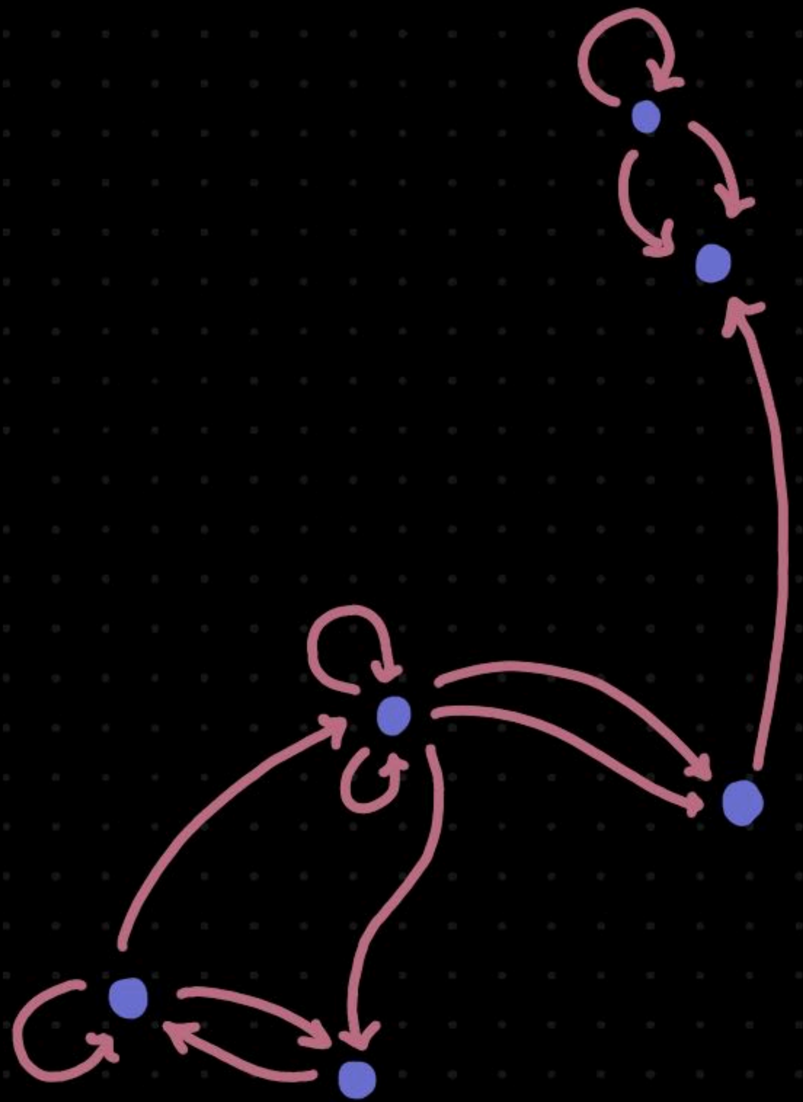


CATEGORIES CONSIST OF:

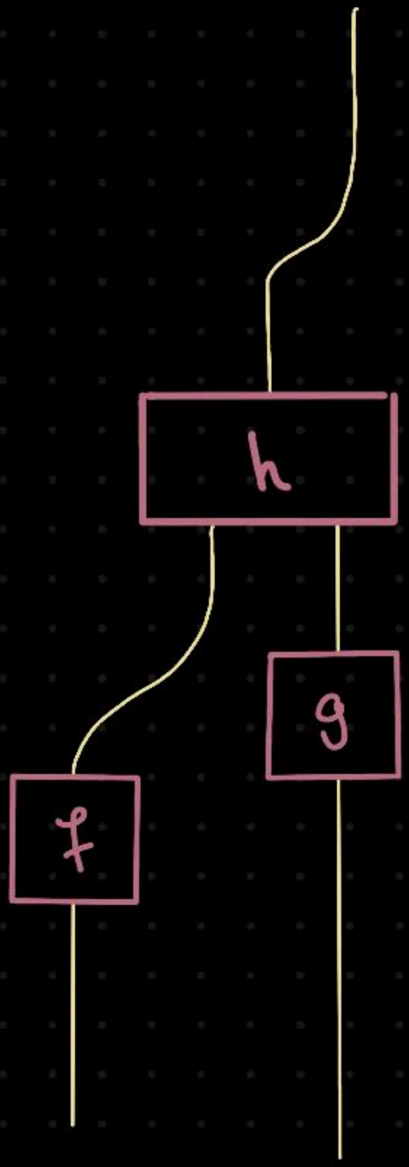
- OBJECTS: H, K ...
- ARROWS: $f : H \longrightarrow K$

WITH IDENTITIES: id_H

AND COMPOSITION: $g \circ f$



PHYSICS AS PROCESSES.



CATEGORIES CONSIST OF:

OBJECTS: H, K ...

ARROWS: $f: H \longrightarrow K$

WITH IDENTITIES: id_H

AND COMPOSITION: $g \circ f$

A MONOIDAL CATEGORY HAS A TENSOR: $\otimes$

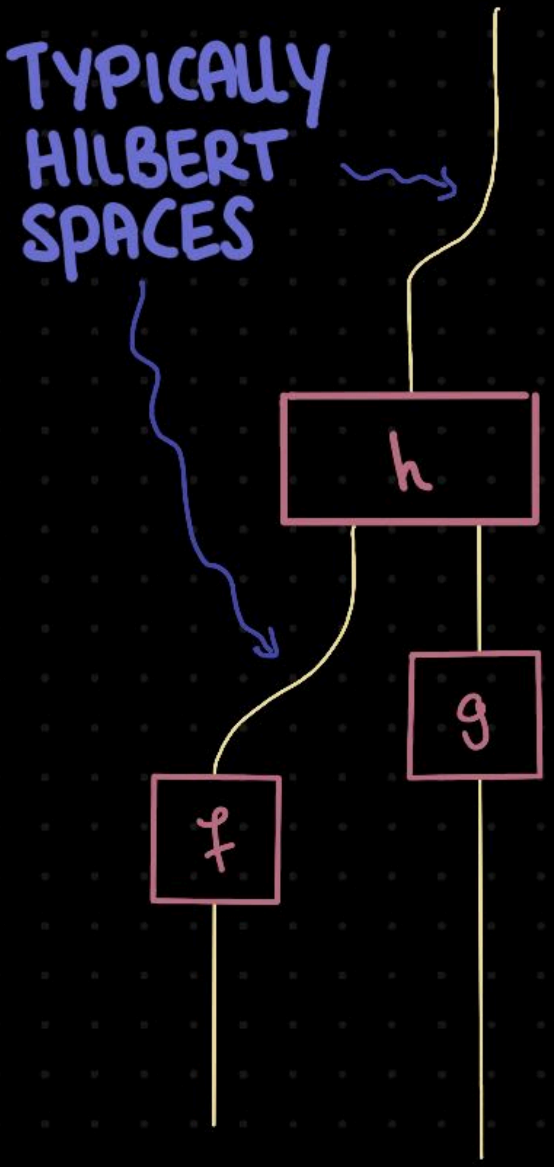
OBJECTS: $H \otimes K$

ARROWS: $f \otimes g : H_1 \otimes K_1 \longrightarrow H_2 \otimes K_2$

AND A TENSOR UNIT: I

$$H \otimes I \cong H \cong I \otimes H$$

PHYSICS AS PROCESSES.



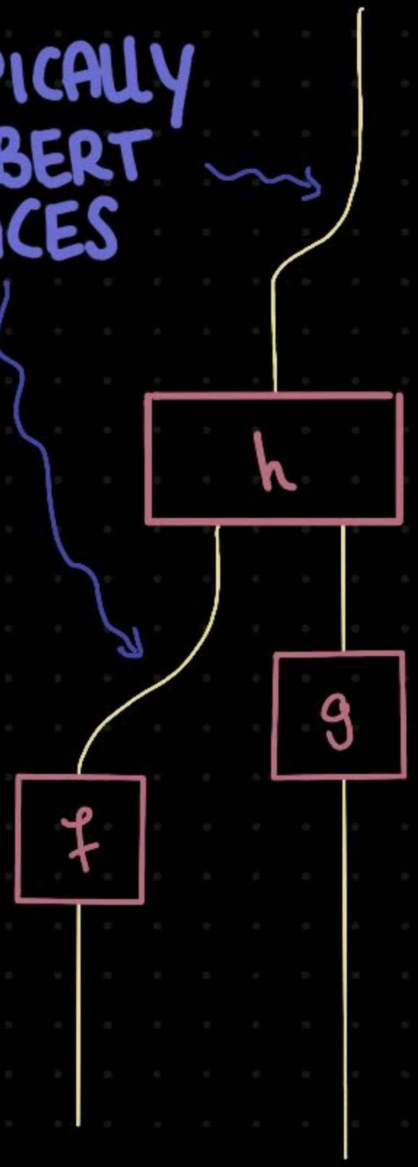
CATEGORICAL QUANTUM MECHANICS: **QM**

HILB
- objects: HILBERT SPACES
- arrows: BOUNDED LINEAR MAPS

CON
- objects: IDEM.
- arrows: LINEAR f s.t.: $\|f\| \leq 1$

PHYSICS AS PROCESSES.

TYPICALLY HILBERT SPACES



HILB IS DAGGER MONOIDAL:

$$H \xrightarrow{f} K \quad \leftrightsquigarrow \quad K \xrightarrow{f^{\dagger}} H$$

INTERACTING NICELY WITH TENSORS:

$$(f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger}$$

BASE FIELD: $I = \mathbb{C}$

$$H \otimes \mathbb{C} \cong H$$

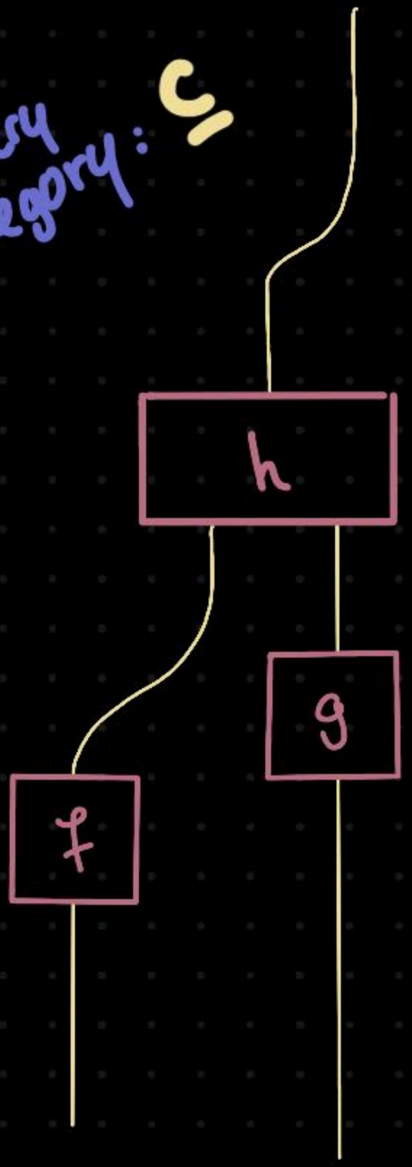
HILB ALSO HAS BIPRODUCTS:

$$H \oplus K \qquad 0 \oplus H \cong H \cong H \oplus 0$$

FORMING A SECOND MONOIDAL STRUCTURE

POSING THE QUESTION.

mystery category: $\underline{C}$



HOW CAN WE TELL OUR PROCESSES ARE QUANTUM ?

WHAT ARE PROPERTIES OF $\underline{C}$ SUCH THAT:

$$\underline{C} \simeq \underline{QM} \ ?$$

(AS DAGGER MONOIDAL CATS.)

ANSWERS?

- HILB: HEUNEN + KORNELL, PNAS 2022 Vol. 119 No. 9, arXiv: 2109.07418.

THIS TALK

- CON: HEUNEN + KORNELL + vdS, arXiv: 2211.02688.

- ONGOING: HILB_A, $\underbrace{\text{FHILB}, \text{UNITARY},}_{\text{HEUNEN + DiMEGLIO}}$ ...
  (NOT BY ME!!)

RECAP OF THE OLD RESULT.

HEUNEN + KORNELL
PNAS 2022 Vol. 119 No. 9
arXiv: 2109.07418.

THE AXIOMS :

(A) $\underline{\mathbf{C}}$ IS DAGGER MONOIDAL

$\dagger : \mathbf{C}^{op} \longrightarrow \mathbf{C}$, id.-on-obj., idempotent, $id_H^\dagger = id_H$.

Monoidal str. whose coherence morphisms are $\dagger$-iso., $f^{-1} = f^\dagger$.

(B) I IS A SIMPLE SEPARATOR

SIMPLE : I has two subobjects.

MON. SEPARATOR : if $\forall\, I \xrightarrow{h} H\ \forall\, I \xrightarrow{k} K$: $f \circ (h \otimes k) = g \circ (h \otimes k)$, then $f = g$.

(C) $\underline{\mathbf{C}}$ HAS $\dagger$-BIPRODUCTS

$\mathbf{C}$ has a zero obj. 0. Coproducts :

$$\begin{tikzcd} H \arrow[dr, "i"'] & & K \arrow[dl, "j"] \\ & H \oplus K & \end{tikzcd}$$

$\dagger$-monomorphisms $\leftarrow$ $f^\dagger \circ f = id$.

$j^\dagger \circ i = 0$

(D) $\underline{\mathbf{C}}$ HAS $\dagger$-EQUALISERS

All equalisers exist, and they are $\dagger$-monomorphisms

(E) $\dagger$-MONOS ARE $\dagger$-KERNELS

Any $\dagger$-mono f is a $\dagger$-kernel, i.e. an equaliser :

$$N \xrightarrow{f} K \underset{0}{\overset{\exists}{\rightrightarrows}} H$$

(F) $\underline{\mathbf{C}}$ HAS DIRECTED COLIMITS OF $\dagger$-MONOS

RECAP OF THE OLD RESULT.

HEUNEN + KORNELL
PNAS 2022 Vol. 119 No. 9
arXiv: 2109.07418.

THE AXIOMS :

(A) $\mathbf{C}$ IS DAGGER MONOIDAL

adjoints & tensors

(B) I IS A SIMPLE SEPARATOR

Base field $\mathbb{C}$ has no nontrivial subspaces
Pure tensors determine morphisms :

$$f(\phi \otimes \psi) = g(\phi \otimes \psi) \implies f = g.$$

(C) $\mathbf{C}$ HAS $\dagger$-BIPRODUCTS

Direct sums of Hilbert spaces

$\dagger$-monos $=$ isometries/closed subspaces

$$H \longrightarrow H \oplus K \longleftarrow K.$$

(D) $\mathbf{C}$ HAS $\dagger$-EQUALISERS

All equalisers exist, and they are $\dagger$-monomorphisms

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RECAP OF THE OLD RESULT.

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THEOREM. IF $\underline{C}$ SATISFIES AXIOMS (A) — (F), THEN:

$$\underline{C} \simeq \underline{\text{HILB}}.$$

(AS DAGGER MONOIDAL CATS.)

RECAP OF THE OLD RESULT.

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THE IDEA: CONSTRUCT

$$\underline{C} \longrightarrow \underline{\mathrm{HILB}}$$
$$H \longmapsto \underline{C}(I, H)$$

WHY? IF H IS A HILBERT SPACE:
$\mathbb{C} \to H$ IS JUST A VECTOR $\psi \in H$
SO:

$$H \cong \underline{\mathrm{HILB}}(\mathbb{C}, H)$$

THE MAGIC: $\underbrace{\text{(A)} - \text{(F)} + \text{SOLÈR'S THEOREM}}$

$\Downarrow$

$\underline{C}(I, H)$ IS A HILBERT SPACE OVER $\underline{C}(I, I) \cong \mathbb{C}$

(or $\mathbb{R}$)

THE STRATEGY.

mystery category: $\underline{\mathbf{D}}$

AXIOMS $\underline{\text{HILB}}$ (KNOWN) $\Longrightarrow$ AXIOMS $\underline{\text{CON}}$

SO WE NEED TO FIGURE OUT:

HOW ARE $\underline{\text{HILB}}$ AND $\underline{\text{CON}}$ RELATED?

THE IDEA:

$$\underline{\mathbf{Hilb}}(H,K)\setminus\{0\} \longleftrightarrow \underline{\mathbf{Con}}(H,K)\setminus\{0\}$$

$$f \longmapsto f/\|f\|$$

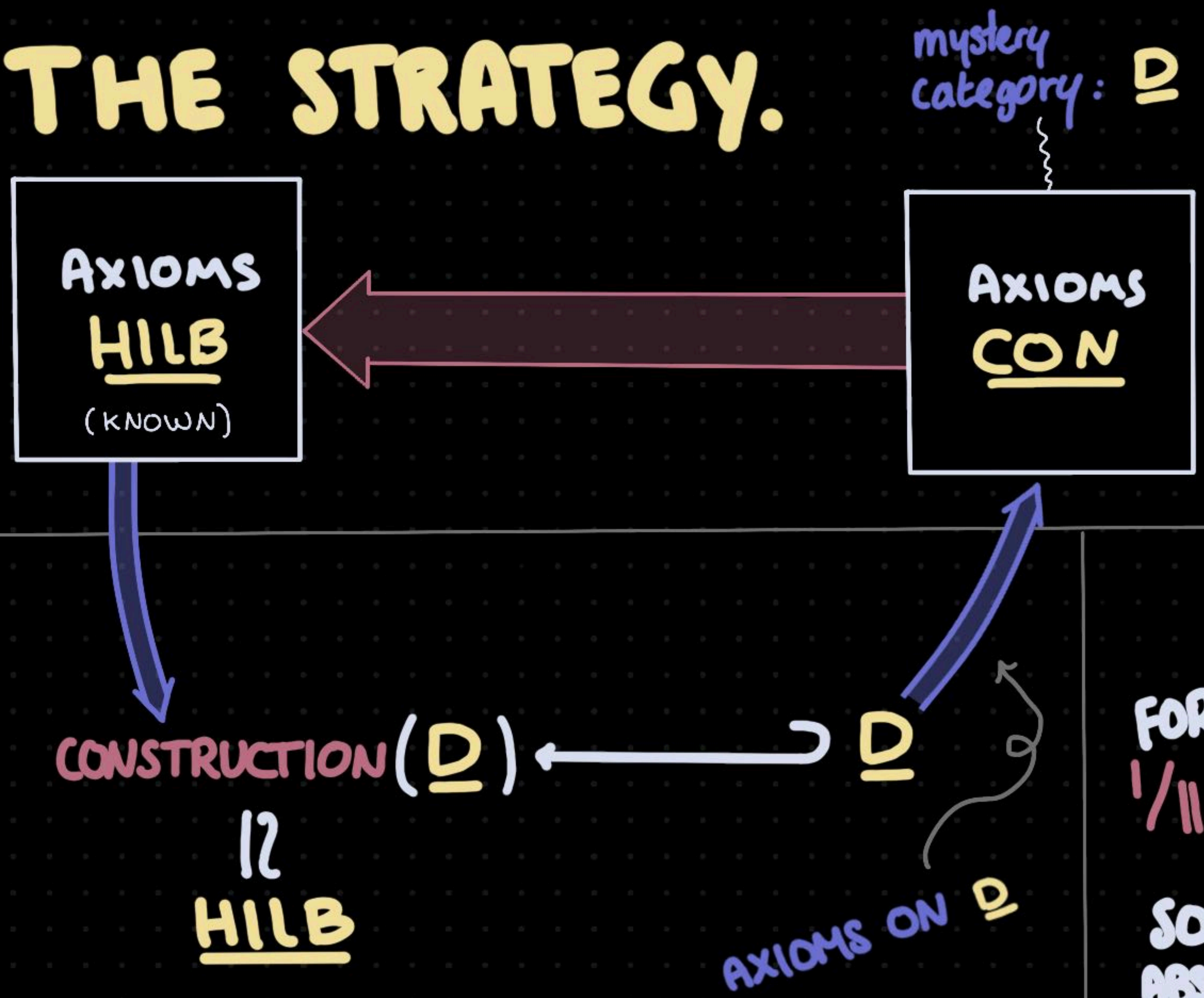
$$f/\|f\| \longleftarrow\!\shortmid \; f$$

THE PROBLEM:

FOR THE "$\longleftarrow\!\shortmid$" DIRECTION, $1/\|f\|$ DOESN'T "EXIST" IN $\underline{\text{CON}}$.

SO HOW DO WE DO THIS ABSTRACTLY IN mystery category $\underline{\mathbf{D}}$?

THE STRATEGY.



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SO HOW DO WE DO THIS ABSTRACTLY IN mystery category $\underline{C}$?

SCALARS.

IN A MONCAT. $\mathcal{C}$ THERE ARE SCALARS:

$$z : I \longrightarrow I$$

THEY FORM A COMMUTATIVE MONOID UNDER COMPOSITION

SCALARS z AND MORPHISMS f CAN BE MULTIPLIED:

$$\begin{tikzcd}
H \arrow[r, "z \bullet f"] \arrow[d, "2"] & K \\
I \otimes H \arrow[r, "z \otimes f"'] & I \otimes K \arrow[u, "2"']
\end{tikzcd}$$

FROM FUNCTORIALITY OF $\otimes$ WE GET:

$$z \bullet (g \circ f)$$
$$\|$$
$$(z \bullet g) \circ f \qquad \text{"SHUFFLING"}$$
$$\|$$
$$g \circ (z \bullet f)$$

SCALARS IN HILB & CON.

WE FIND:

$\mathbb{C}$

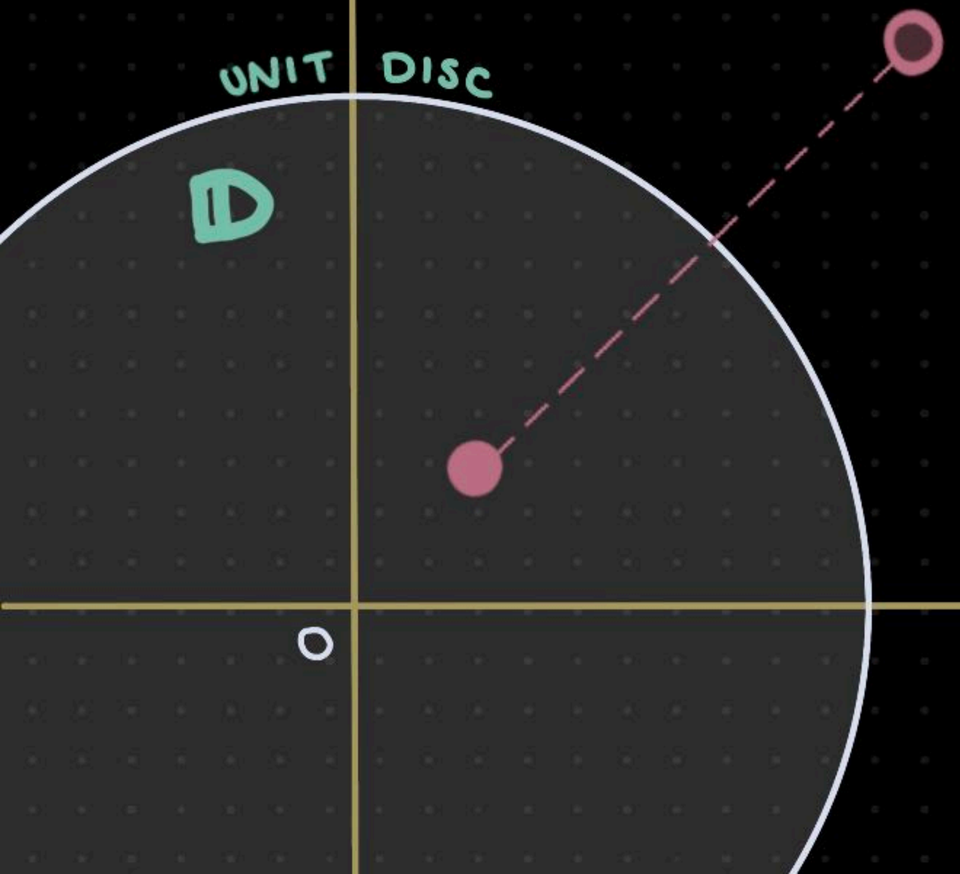
complex numbers

$\underline{\mathbf{HILB}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}$ AND $\underline{\mathbf{CON}}(\mathbb{C},\mathbb{C}) \cong \mathbb{D}$.

THE IDEA: CONSTRUCTION(—) ADDS FORMAL INVERSES FOR ALL SCALARS.

THEN: CONSTRUCTION($\underline{\mathbf{CON}}$) $\cong$ $\underline{\mathbf{HILB}}$.

ROUGHLY: id.-on-obj., and send formal inverses to ACTUAL inverse in $\underline{\mathbf{HILB}}$



THE NEW AXIOMS.

ALMOST BIPRODUCTS

(1) $\mathbf{D}$ IS A †-CAT.

(2) $\mathbf{D}$ IS A †-RIG CAT.

Two †-monoidal structures: $(\otimes, I)$ and $(\oplus, 0)$ such that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$ and $\otimes$ distributes over $\oplus$.

(3) $(\oplus, 0)$ IS AFFINE

0 is initial, and hence a zero obj. This gives natural:

$$inl_{HK} : H \longrightarrow H \oplus K$$
$$inr_{HK} : K \longrightarrow H \oplus K$$

(4) inl, inr ARE JOINTLY EPIC

$$\left.\begin{array}{l} f \circ inl = g \circ inl \\ f \circ inr = g \circ inr \end{array}\right\} \Longrightarrow f = g$$

(5) THERE IS MIXTURE

$\exists\, s : I \longrightarrow I \oplus I$ with $inl^\dagger \circ s \neq 0 \neq inr^\dagger \circ s$

UNIT AXIOMS

(6) I IS †-SIMPLE

(7) I IS A $\otimes$-SEPARATOR

†-AXIOMS

(8) $\mathbf{D}$ HAS ALL †-EQUALISERS

(9) †-MONOS ARE †-KERNELS

(10) SUBOBJECTS ARE DETERMINED BY POSITIVE MAPS

$s = t$ as subobj. iff $s \circ s^\dagger = t \circ t^\dagger$

(11) $\mathbf{D}$ HAS ALL DIRECTED COLIMITS

TOWARDS THE CONSTRUCTION.

SOME LEMMAS:

(†-AXIOMS)

LEMMA. MORPHISMS FACTOR AS

$$H \xrightarrow{\text{epi}} E \overset{\dagger\text{-mono}}{\rightarrowtail} K$$

(UNIT AXIOMS)

LEMMA. IF $z \neq 0$, THEN $z \cdot f = z \cdot g$ IMPLIES $f = g$

IN HILB:

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, equal] & & K \arrow[d, equal] \\ H \arrow[r, "f"'] & \mathrm{im}(f) \arrow[r, hook] & K \end{tikzcd}$$

SCALAR MULTIPLICATION BY $z \in \mathbb{C}$, $z \neq 0$ IS CANCELLATIVE

THE CONSTRUCTION.

GIVEN $\begin{cases} \text{MON CAT} : \underline{D} \\ \text{SCALARS} : \mathbb{D} := \underline{D}(I, I) \end{cases}$

THERE IS A CAT.

$$\text{CONSTRUCTION}(\underline{D}) = \underline{D}(\mathbb{D}^{-1})$$

WITH:

OBJECTS: SAME AS $\underline{D}$

ARROWS: OF THE FORM

$$H \xrightarrow{[t/z]} K$$

WHERE $t \in \underline{D}(H, K)$
$z \in \mathbb{D} \setminus \{0\}$

UNDER EQUIVALENCE RELATION:

$$(t/z) \sim (t'/z') \iff z' \bullet t = z \bullet t'$$

(TRANSITIVE BY LEMMA.)

IDENTITY: $[id_H/1]$

COMPOSITION:

$$[t/z] \circ [s/w] := [t \circ s / z \bullet w]$$

(WELL DEFINED BY SHUFFLING & MONIC SCALARS.)

THE CONSTRUCTION.

THE CONSTRUCTION IS A LOCASISATION:

WE HAVE A FAITHFUL INCLUSION FUNCTOR:

$$\begin{tikzcd} \underline{D} \arrow[r, hook] & \underline{D}(\mathbb{D}^{-1}) \\ f \arrow[r, mapsto] & {[f/1]} \end{tikzcd}$$

PROPOSITION. IF F IS STRONG $\otimes$-MONOIDAL AND $\forall z \in \mathbb{D}\setminus\{0\}$: $F(z)$ INVERTIBLE:

$$\begin{tikzcd} \underline{D} \arrow[r, hook] \arrow[dr, "F"'] & \underline{D}(\mathbb{D}^{-1}) \arrow[d, dashed, "\exists!\ \text{STRONG } \otimes\text{-MONOIDAL}"] \\ & \underline{C} \end{tikzcd}$$

THE CONSTRUCTION.

AFTER A LOT OF DETAILS... :

THEOREM. IF $\underline{D}$ SATISFIES AXIOMS (I) — (II) THEN $\underline{D}(\mathbb{D}^{-1})$ SATISFIES AXIOMS (A) - (F), SO:

$$\underline{D} \hookrightarrow \underline{D}(\mathbb{D}^{-1}) \cong \underline{\mathrm{HILB}}$$

TOWARDS THE THEOREM.

HENCE $\underline{D} \subseteq \underline{HILB}$.
WE ARE "JUST" LEFT TO SHOW:
$$\underline{D} = \underline{CON}.$$

$\underline{D}(\mathbb{D}^{-1})$ HAS DAGGER $[f/z]^{\dagger} := [f^{\dagger}/z^{\dagger}]$.

LEMMA. SUPPOSE $t^{\dagger} \circ t = z^{\dagger} \bullet z \bullet id$, FOR $z \in \mathbb{D}$. THEN:
$\exists\, \dagger$-MONO $m : t = z \bullet m$.

proof. $t : \bullet \xrightarrow{e} \bullet \overset{m}{\rightarrowtail} \bullet$

THEN $e^{\dagger} \circ e = (z \bullet id)^{\dagger} \circ (z \bullet id)$
SO (10) GIVES
$\exists\, \dagger$-iso. $u : e = z \bullet u$

POSITIVE MAPS DETERMINE SUBOBJECTS

AND HENCE:
$$t = z \bullet \underbrace{(m \circ u)}_{\dagger\text{-mono.}}.$$

PROPOSITION. THERE IS AN ISO.:
$$\underline{D}_{\dagger m} \overset{\sim}{\hookrightarrow} \underline{D}(\mathbb{D}^{-1})_{\dagger m}$$
$\dagger$-monics

proof. WE JUST NEED TO SHOW THE INCLUSION FUNCTOR IS <u>FULL</u>. SO LET $[t/z]$ BE $\dagger$-MONO.
THIS MEANS:
$$t^{\dagger} \circ t = z^{\dagger} \bullet z \bullet id.$$
HENCE $t = z \bullet m$ AND
$$m \longmapsto [m/1] = [t/z].$$

TOWARDS THE THEOREM.

HENCE $\mathbf{D} \subseteq \mathbf{HILB}$.
WE ARE "JUST" LEFT TO SHOW:
$\mathbf{D} = \mathbf{CON}$.

PROPOSITION. $\mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$.

actual complex numbers

Proof. FOR FINITE $R \subseteq \mathbb{N}$ DEFINE:

$$I^R := \underbrace{I \oplus \cdots \oplus I}_{|R|\text{-times}}.$$

WE GET COLIMIT IN $\mathbf{HILB}$:

$$\begin{tikzcd}
\cdots \; I^R \arrow[rr, tail, "i_{RS}", "R \subseteq S"'] \arrow[dr, bend right, "i_R"'] & & I^S \; \cdots \arrow[dl, bend left, "i_S"] \\
& I^{\mathbb{N}} &
\end{tikzcd}$$

$\dagger$-MONOS ARE THE SAME IN $\mathbf{HILB}$ & $\mathbf{D}$.

SO WE GET COLIMIT IN $\mathbf{D}$: $j_R : I^R \rightarrowtail H$.

FOR $z \in \mathbb{D}$, CONSTRUCT NATURAL

$$t_R : I^R \longrightarrow I^R,$$

$$t_{\{n\}} : I \longrightarrow I$$
$$s \longmapsto z^n \cdot s$$

$$\begin{tikzcd}
I \arrow[r, tail, "j_{\{n\}}"] \arrow[d, "t_{\{n\}}"'] & H \arrow[d, "t"] \\
I \arrow[r, tail, "j_{\{n\}}"'] & H
\end{tikzcd}$$

HENCE t HAS EIGENVALUE z^n, $\forall n$. SO:

$$\|t\| \geq |z|^n.$$

SINCE t IS BOUNDED:

$$|z| \leq 1.$$

TOWARDS THE THEOREM.

HENCE $\underline{\mathbf{D}} \subseteq \underline{\mathbf{HILB}}$.
WE ARE "JUST" LEFT TO SHOW:

$$\underline{\mathbf{D}} = \underline{\mathbf{CON}}.$$

<u>LEMMA</u>. WE HAVE: $\underline{\mathbf{D}}(H,K) \subseteq \{ t \in \underline{\mathbf{HILB}}(H,K) : \|t\| \leqslant 1 \}$.

proof. LET $t \in \underline{\mathbf{D}}(H,K)$. SFTSOC $\|t\| > 1$.

THEN $\exists$ UNIT VECTOR $x : I \longrightarrow H$, $\|t \circ x\| > 1$.

x IS $\dagger$-MONO, SO $x \in \underline{\mathbf{D}}(I,H)$. BUT THEN:

$$\mathbb{D} \ni x^{\dagger} \circ t^{\dagger} \circ t \circ x = \|t \circ x\|^2 > 1. \quad ⚡$$

<u>CONCLUSION</u>: $\underline{\mathbf{D}} \hookrightarrow \underline{\mathbf{D}}(\mathbb{D}^{-1}) \simeq \underline{\mathbf{HILB}}$ LANDS IN $\underline{\mathbf{CON}}$.

TOWARDS THE THEOREM.

HENCE $\underline{\mathbf{D}} \subseteq \underline{\mathbf{HILB}}$.
WE ARE "JUST" LEFT TO SHOW:
$$\underline{\mathbf{D}} = \underline{\mathbf{CON}}.$$

LEMMA. WE HAVE: $\underline{\mathbf{D}}(H,K) \supseteq \{ t \in \underline{\mathbf{HILB}}(H,K) : \|t\| \leq 1 \}$.

proof. FIRST: $t : H \to H$, $\|t\| < 1$. BY RUSSO-DYE-GARDNER:

$$t = \frac{1}{n}(u_1 + \cdots + u_n)$$

$\dagger$-isomorphisms in $\underline{\text{HILB}}$

THE NORMALISED DIAGONAL: $\omega : H \longrightarrow H \oplus \cdots \oplus H$
IS $\dagger$-MONO IN $\underline{\mathbf{HILB}}$, AND: $x \longmapsto (x, \cdots x)/\sqrt{n}$

$$t = \underbrace{\omega^\dagger \circ (u_1 \oplus \cdots \oplus u_n) \circ \omega}_{\text{IN } \underline{\mathbf{D}}}$$

IN GENERAL: POLAR DECOMPOSITION IN $\underline{\mathbf{HILB}}$:

$$t = \underbrace{u \circ v^\dagger}_{\dagger\text{-monos}} \circ \underbrace{s}_{\text{previous case}}$$

TOWARDS THE THEOREM.

HENCE $\underline{D} \subseteq \underline{HILB}$.
WE ARE "JUST" LEFT TO SHOW:
$$\underline{D} = \underline{CON}.$$

LEMMA. WE HAVE: $\underline{D}(H,K) \supseteq \{ t \in \underline{HILB}(H,K) : \|t\| \leq 1 \}$.

proof. LASTLY: $\|t\| = 1$. TAKE

$$0 < z_1 < z_2 < \cdots < 1, \quad \sup z_n = 1.$$

THEN $\|z_n \bullet t\| < 1$, SO $z_n \bullet t \in \underline{D}(H,K)$.

IN $\underline{D}$:

COLIMIT IN $\underline{D}$

$$\begin{tikzcd}
H \arrow[r, "z_1/z_2"] \arrow[drrr, "z_1"] \arrow[ddrrr, bend right, "z_1 \bullet t"'] & H \arrow[r, "z_2/z_3"] \arrow[drr, "z_2"] \arrow[ddrr, bend right, "z_2 \bullet t"'] & H \arrow[r] \arrow[dr, "z_3"] \arrow[ddr, bend right, "z_3 \bullet t"'] & \cdots \\
 & & & H \arrow[d, "\exists ! S"] \\
 & & & K
\end{tikzcd}$$

HENCE WE GET IN PARTICULAR:

$$\begin{aligned}
& z_1 \bullet S \\
&= \quad (\text{SHUFFLING}) \\
& S \circ (z_1 \bullet \mathrm{id}) \\
&= \quad (\text{DIAGRAM}) \\
& z_1 \bullet t
\end{aligned}$$

AND SINCE $z_1 \neq 0$:

$$t = S \in \underline{D}(H,K).$$

THE THEOREM.

THEOREM. IF $\underline{\mathbf{D}}$ SATISFIES AXIOMS (I) — (II), THEN:

$$\underline{\mathbf{D}} \simeq \underline{\mathbf{CON}} .$$

(AS DAGGER RIG CATS.)

THE THEOREM.

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Thanks very much for your attention!