Ordered Locales

Edinburgh Category Theory Seminar

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joint work with Chris Heunen

8th November 2023

{arXiv: 2303.03813}

Overview

Locales

Ordered locales

Adjunction

Future work

Motivation

tensor topology

Spacetimes

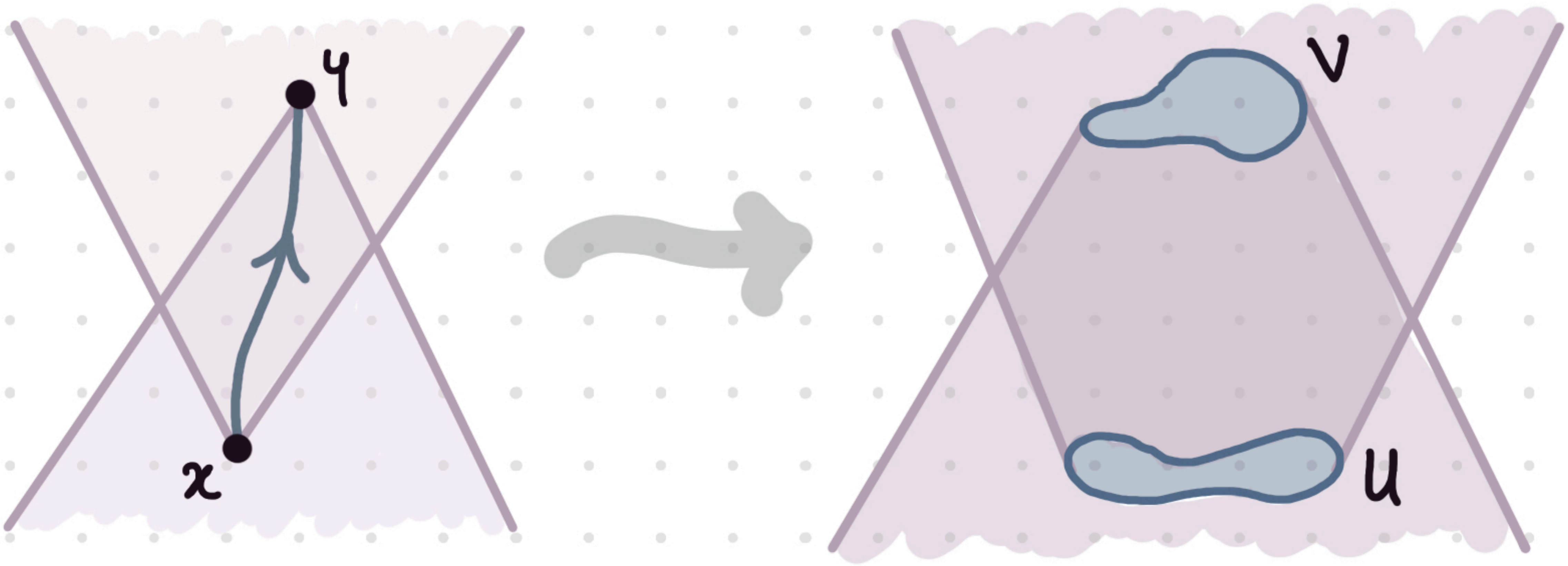
generalising Stone duality

Idea

$$
\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[u, hook] \arrow[l, shift left, "\mathrm{pt}"]
\end{tikzcd}
$$

Causality + spaces

Idea



causal order

$x \leq y$

region causality

$U \trianglelefteq V$

$$
\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
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\end{tikzcd}
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$$\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left=2] \\
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\end{tikzcd}$$

Frames

FRAMES

Complete lattice L with:

$$u \wedge \bigvee v_i = \bigvee (u \wedge v_i)$$

MAPS

functions $h : L \longrightarrow M$:

$$h(\top) = \top$$

$$h(u \wedge v) = h(u) \wedge h(v)$$

$$h\left(\bigvee u_i\right) = \bigvee h(u_i)$$

We get a category:

Frm

EXAMPLES

- c. Heyting alg.
- c. Boolean alg.
- measure space
- topologies
- $Sub_{\mathcal{E}}(1)$

Locales

$$\begin{aligned} \mathbf{Top} &\longrightarrow \mathbf{Frm}^{op} \\ S &\longmapsto \mathcal{O}S \\ (S \xrightarrow{f} T) &\longmapsto (\mathcal{O}T \xrightarrow{f^{-1}} \mathcal{O}S) \end{aligned}$$

Frames as algebraic dual to a type of space:

$$\mathbf{Loc} := (\mathbf{Frm})^{op}$$

LOCALES

object: $X \in \mathbf{Loc} \longleftrightarrow \mathcal{O}X \in \mathbf{Frm}$

arrows: $(X \xrightarrow{f} Y) \in \mathbf{Loc} \longleftrightarrow (\mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X) \in \mathbf{Frm}$

Locales

$$\mathbf{Loc} \xrightarrow{\ ?\ } \mathbf{Top}$$

Not every locale comes from a topological space!! Yet:

POINT

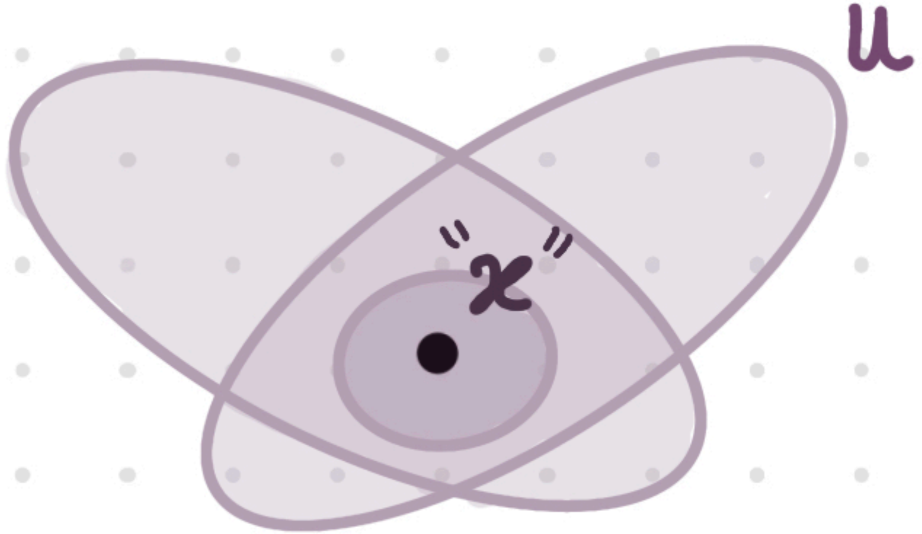
map of locales:

$$p : 1 \longrightarrow X$$

$$\mathcal{O}X \xrightarrow{p^{-1}} \mathcal{O}1 := \{F < T\}$$

$$\Updownarrow$$

$$\mathcal{F} = \{ U \in \mathcal{O}X : p^{-1}(U) = T \}$$



$$U \in \mathcal{F} \iff \text{"}x\text{"} \in \mathcal{F}$$

Locales

$$\mathbf{Loc} \xrightarrow{\;?\;} \mathbf{Top}$$

Not every locale comes from a topological space!! Yet:

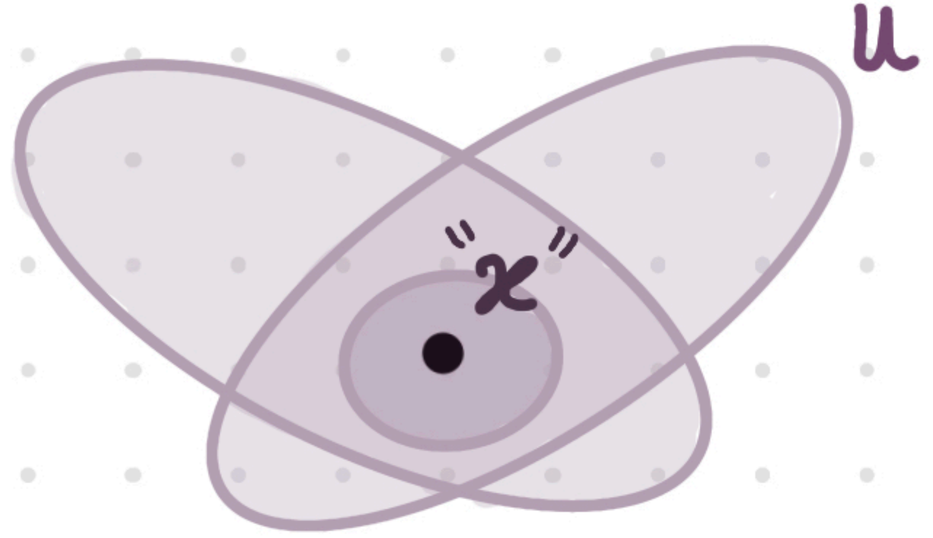
C.PRIME FILTER

A filter $\mathcal{F} \subseteq \mathcal{O}X$:

- $X \in \mathcal{F}$;
- $U, V \in \mathcal{F} \Longrightarrow U \wedge V \in \mathcal{F}$;
- $U \in \mathcal{F},\ U \subseteq V \Longrightarrow V \in \mathcal{F}$;

Such that:

- $\emptyset \notin \mathcal{F}$;
- $\bigvee U_i \in \mathcal{F} \Longrightarrow \exists i : U_i \in \mathcal{F}$.



$$U \in \mathcal{F} \Longleftrightarrow \text{"}x\text{"} \in \mathcal{F}$$

Locales

OPENS

basic opens for $U \in \mathcal{O}X$:

$$pt(U) := \{ \mathcal{F} : U \in \mathcal{F} \}$$

$$x : x \in U$$

$$\mathbf{Loc} \xrightarrow{pt} \mathbf{Top}$$

$$X \longmapsto pt(X)$$

$$f \longmapsto pt(f)$$

MAPS

for $f : X \longrightarrow Y$:

$$pt(f)(\mathcal{F}) = \{ V \in \mathcal{O}Y : f^{-1}(V) \in \mathcal{F} \}$$

$$f(x) \qquad\qquad V : x \in f^{-1}(V)$$

Locales

THEOREM

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left=2, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, shift left=2, "\mathrm{pt}"] \end{tikzcd}$$

Locales

UNIT

$$\eta_S : S \longrightarrow \mathrm{pt}\,\mathrm{Loc}(S)$$
$$x \longmapsto \mathcal{F}_x := \{U \in \mathcal{O}S : x \in U\}$$

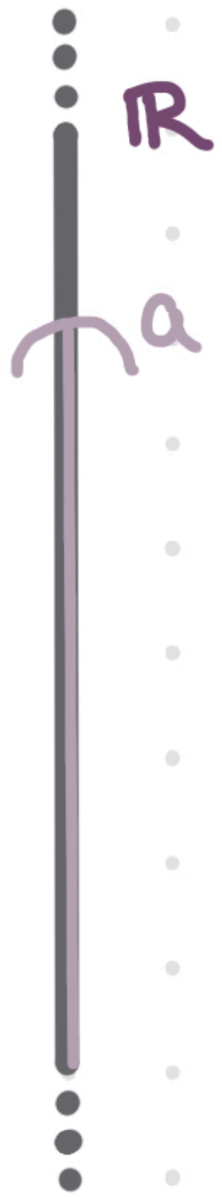
SOBER

S is a fixed point iff it:

- is T_0; $\quad \mathcal{F}_x = \mathcal{F}_y \implies x = y$
- has enough points. $\quad \forall \mathcal{F}\, \exists x : \mathcal{F} = \mathcal{F}_x$

E.G. $\overleftarrow{\mathbb{R}}$ with basis opens $(-\infty, a)$.

Then $\mathrm{pt}(\overleftarrow{\mathbb{R}}) \cong \mathbb{R} \cup \{-\infty\}$.



Locales

COUNIT

$$\varepsilon_X : \mathrm{Loc}\, pt(X) \longrightarrow X$$

$$\begin{aligned} \varepsilon_X^{-1} : \mathcal{O}X &\longrightarrow \mathcal{O}pt(X) \\ U &\longmapsto pt(U) \end{aligned}$$

SPATIAL

X is a fixed point iff:

$$pt(U) = pt(V) \Longrightarrow U = V$$

Locales

Start with $S \in \mathbf{Top}$.

REG. O.

$$\mathcal{R}S := \{ U \in \mathcal{O}S \ : \ U = (\overline{U})^{\circ} \}$$

defines $\mathrm{Reg}(S) \in \mathbf{Loc}$.

If S is Hausdorff:

LEMMA

$$\mathrm{Pt}\,\mathrm{Reg}(S) \cong \{ \text{isolated points in } S \}$$

E.G. $\mathrm{Pt}\,\mathrm{Reg}(\mathbb{R}^n) = \emptyset$

$$
\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left=1.5] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[u, hook] \arrow[l, shift left=1.5]
\end{tikzcd}
$$

Ordered Spaces

ORD. SP. A space S with a preorder $\leqslant$.

MAPS continuous monotone functions $f : S \longrightarrow T$.

$x \leqslant y \implies f(x) \leqslant f(y)$.

We get a category:

OrdTop

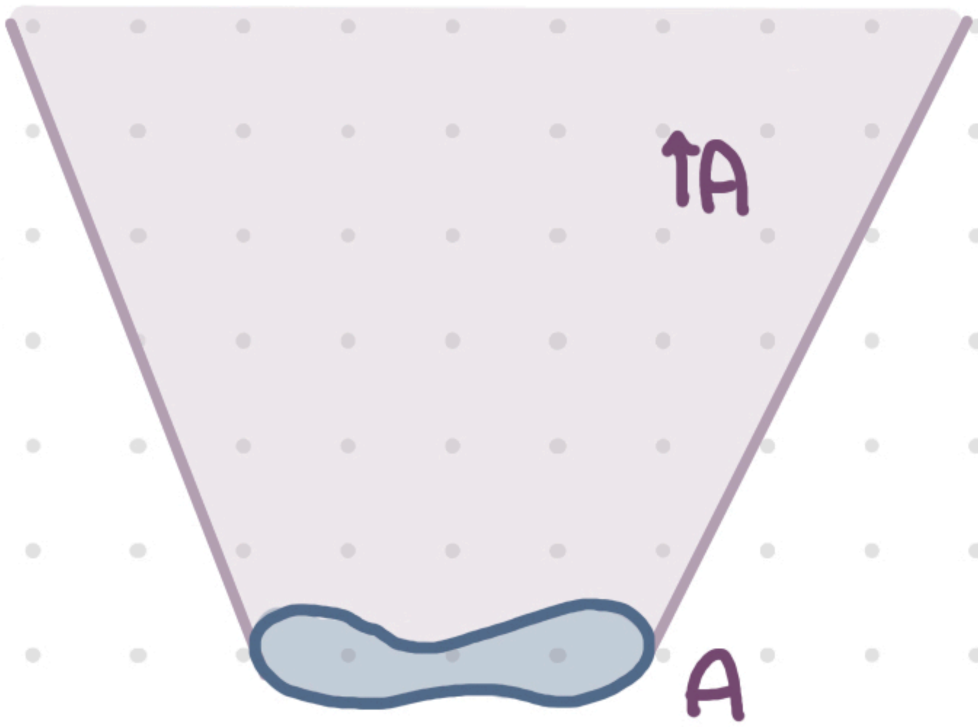
Ordered Spaces

capture $\leqslant$ in terms of open regions?

CONES

the future cone of $A \subseteq S$:

$\uparrow A := \{ x \in S \mid \exists a \in A : a \leqslant x \}$.



LEMMA

- $A \subseteq B \implies \uparrow A \subseteq \uparrow B$
- $A \subseteq \uparrow A$
- $\uparrow\uparrow A \subseteq \uparrow A$

$x \leqslant y \iff y \in \uparrow\{x\}$

Ordered Spaces

capture $\leqslant$ in terms of open regions?

CONES

the future cone of $A \subseteq S$:

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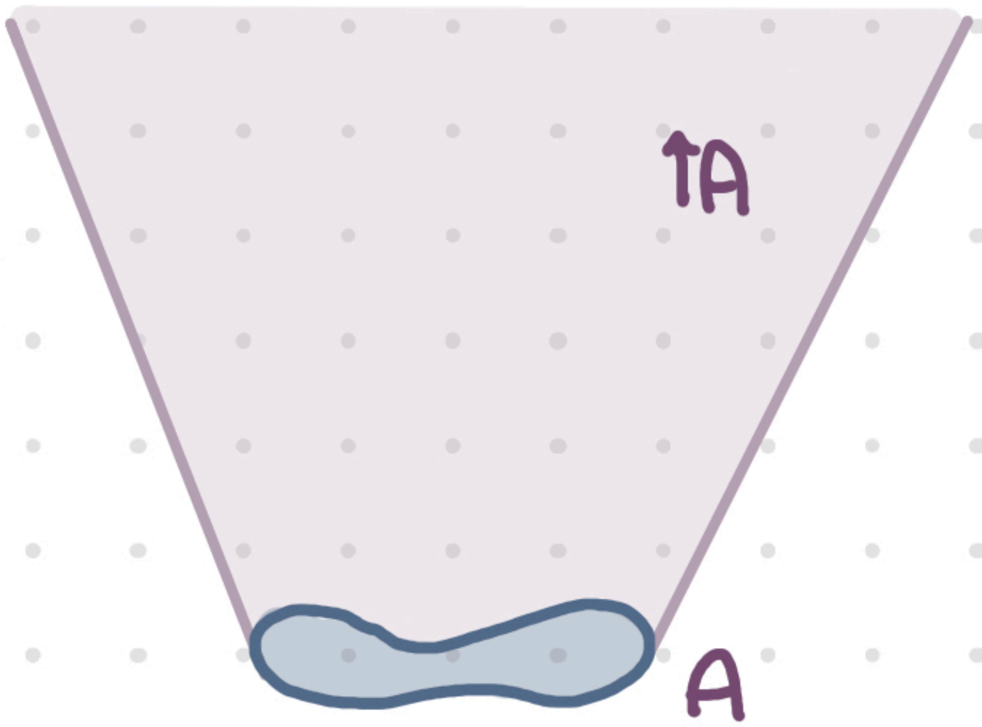


LEMMA

$f : S \longrightarrow T$ monotone iff:

$\uparrow f^{-1}(B) \subseteq f^{-1}(\uparrow B)$.

Only if: $f^{-1}(B) \ni x \leqslant y \implies B \ni f(x) \leqslant f(y) \implies y \in f^{-1}(\uparrow B)$.

if: $x \leqslant y \implies y \in \uparrow\{x\} \subseteq \uparrow g^{-1}\{g(x)\} \subseteq g^{-1}(\uparrow\{g(x)\}) \implies g(x) \leqslant g(y)$

Ordered Spaces

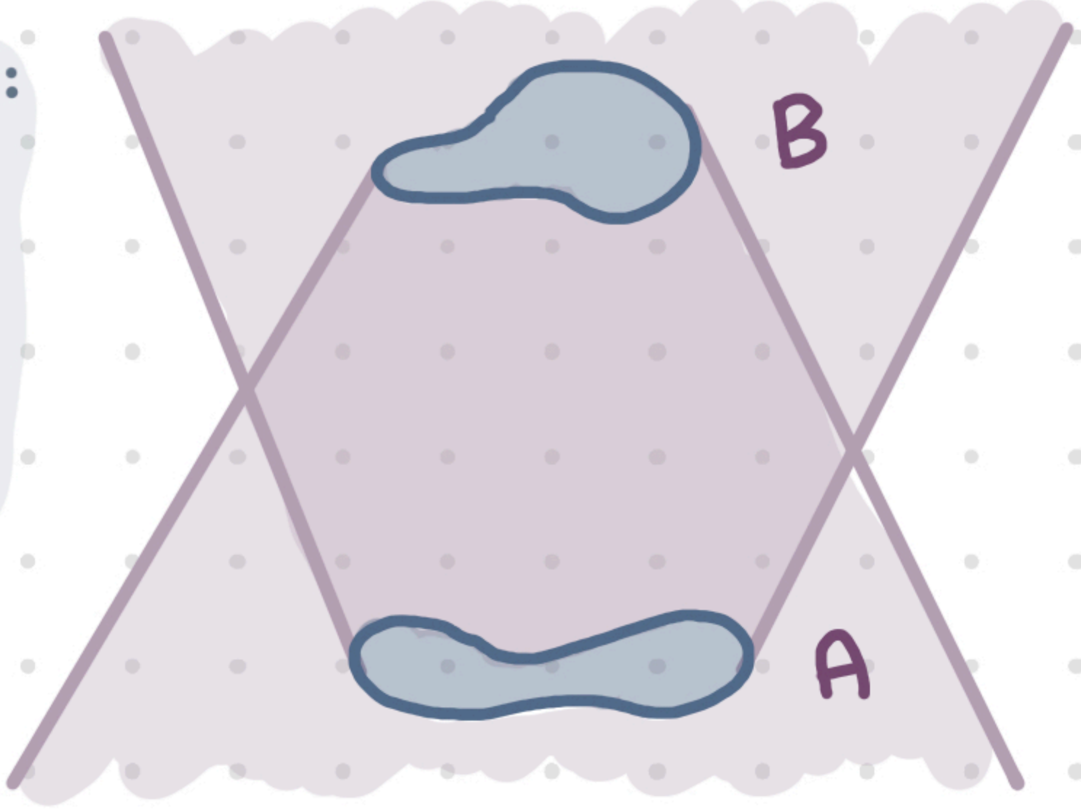
capture $\leqslant$ in terms of open regions?

EM ORDER

If $(S, \leqslant)$ is a preorder:

$$A \trianglelefteq B \iff \begin{cases} A \subseteq \downarrow B, \\ B \subseteq \uparrow A \end{cases}$$

is a preorder on $\mathcal{P}(S)$.



$(\mathrm{Loc}(S), \trianglelefteq)$ as prototypical ordered locale

$$
\begin{tikzcd}
\mathrm{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{OrdLoc} \arrow[l, shift left] \\
\mathrm{Top} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Loc} \arrow[l, shift left] \arrow[u, hook]
\end{tikzcd}
$$

Ordered Locales

ORD. LOC.

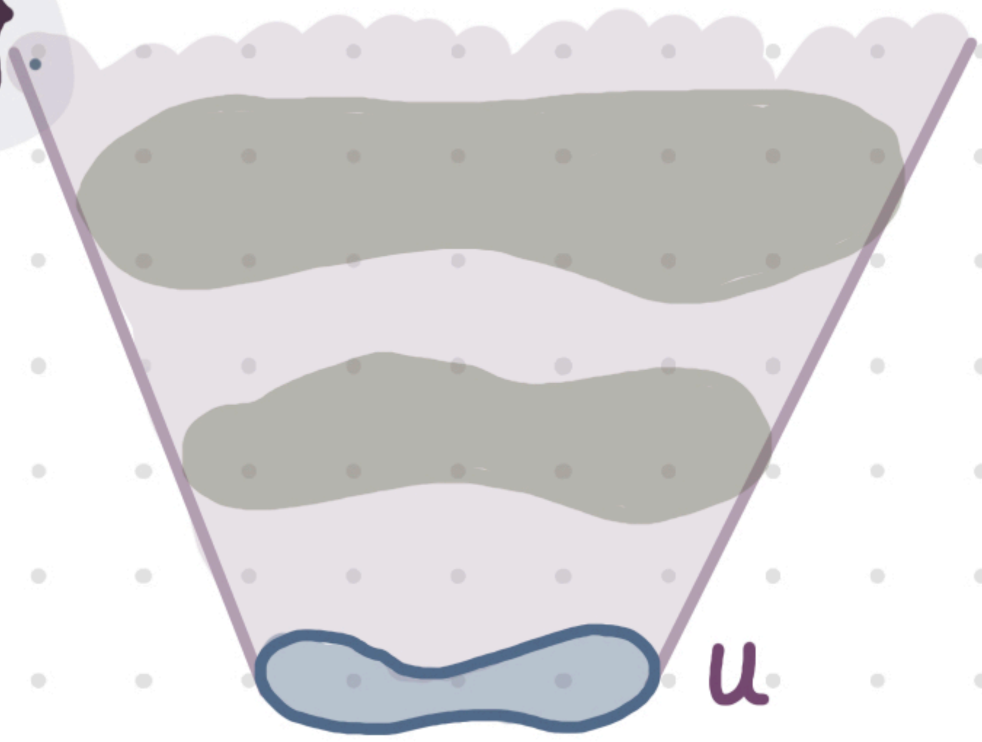
a locale X with preorder $\trianglelefteq$ on $\mathcal{O}X$:

$$\forall i: U_i \trianglelefteq V_i \implies \bigvee U_i \trianglelefteq \bigvee V_i .$$

Ordered Locales

CONES

$$\uparrow U := \bigvee \{ W \in \mathcal{O}X : U \trianglelefteq W \}$$



LEMMA

- $\downarrow U \trianglelefteq U \trianglelefteq \uparrow U$
- $U \subseteq V \Longrightarrow \uparrow U \subseteq \uparrow V$
- $U \subseteq \uparrow U$
- $\uparrow\uparrow U \subseteq \uparrow U$

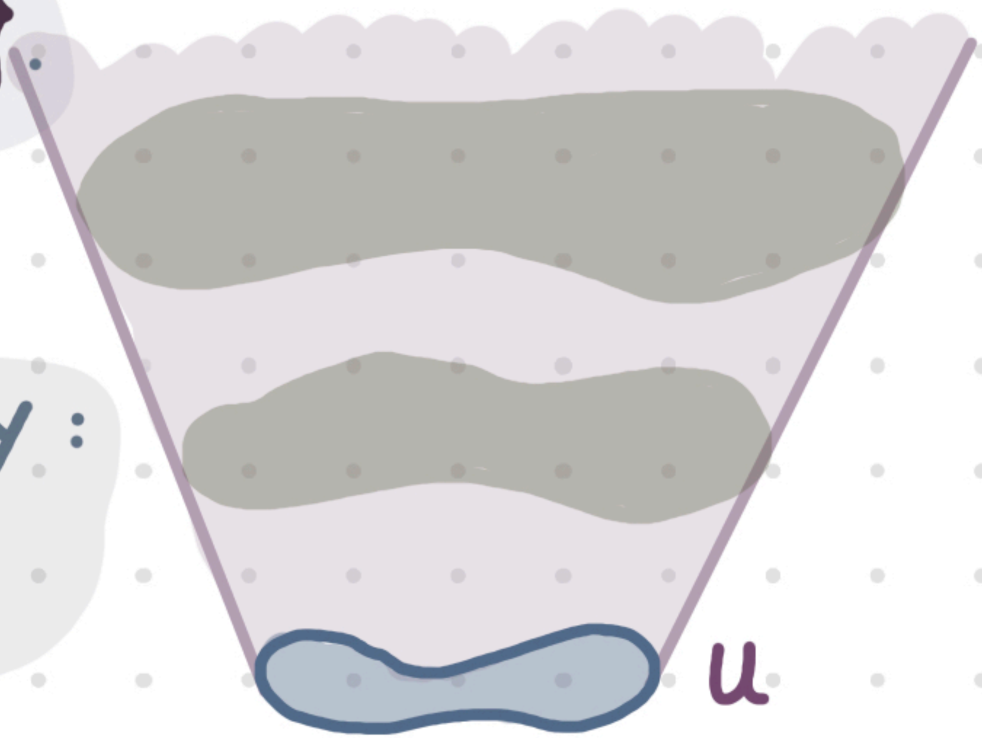
$$U \trianglelefteq W,\ V \trianglelefteq V \Rightarrow V = U \vee V \trianglelefteq V \vee W$$

$$\Rightarrow W \subseteq V \vee W \subseteq \uparrow V.$$

Ordered Locales

CONES

$$\uparrow U := \bigvee\{W \in \mathcal{O}X : U \triangleleft W\}$$



MAPS

monotone map $f : X \longrightarrow Y$:

$$\uparrow f^{-1}(U) \sqsubseteq f^{-1}(\uparrow U).$$

Ordered Locales

CONES

$$\Uparrow U := \bigvee \{ W \in \mathcal{O}X : U \trianglelefteq W \}.$$

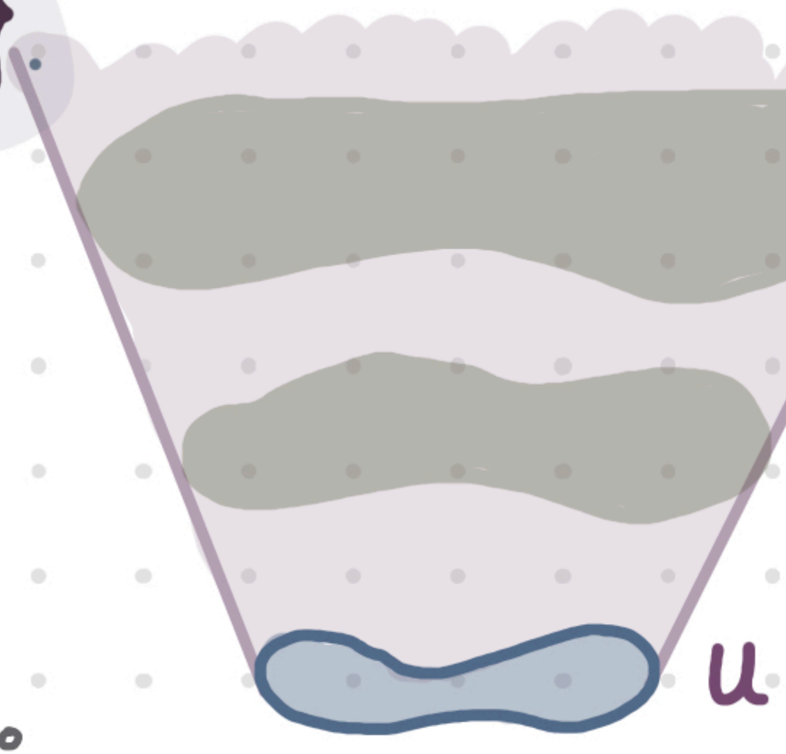
E.G.

in a space $(S, \leq)$:

$$\Uparrow U = (\uparrow U)^\circ , \quad \Downarrow U = (\downarrow U)^\circ$$

$$x \in \Uparrow U \Rightarrow \exists W \ni x : U \trianglelefteq W \Rightarrow x \in W \subseteq \Uparrow U.$$

$$(\uparrow U)^\circ \subseteq \uparrow U, \; U \subseteq (\uparrow U)^\circ \subseteq \downarrow (\uparrow U)^\circ \Longrightarrow U \trianglelefteq (\uparrow U)^\circ.$$



Adjunction

$$\begin{tikzcd} \mathbf{OrdTop} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left] \end{tikzcd}$$

$$(S, \leqslant) \longmapsto (\mathrm{Loc}(S), ?)$$

$$(\mathrm{pt}(X), ?) \longmapsfrom (X, \trianglelefteq)$$

Adjunction

$$
\begin{tikzcd}
\mathbf{OrdTop} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left=1.5ex]
\end{tikzcd}
$$

$$(S, \leq) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$$

Egli-Milner

$$(\mathrm{pt}(X), \boxed{?}) \longmapsfrom (X, \trianglelefteq)$$

Adjunction

OrdTop $\longrightarrow$ **OrdLoc**

$(S, \leqslant) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$

OPEN CONES

$(S,\leqslant)$ has OC if:
$\forall U \in \mathcal{O}S : \uparrow U, \downarrow U \in \mathcal{O}S$

only if: OC $\Rightarrow \uparrow U = \Uparrow U$, $\downarrow U = \Downarrow U$

LEMMA

S has OC iff:
$T \xrightarrow{g} S$ monotone $\Longrightarrow$
$\mathrm{Loc}(T) \xrightarrow{\mathrm{Loc}(g)} \mathrm{Loc}(S)$ monotone

if: we need $\Uparrow U \subseteq (\uparrow U)^\circ$.

$y \in \Uparrow U \Longrightarrow \exists x \in U : x \leqslant y$

$\Longrightarrow \{0<1\} \xrightarrow{g} S : \begin{array}{l} g(0) = x \\ g(1) = y \end{array}$

$\Longrightarrow \mathrm{Loc}(g)$ monotone

$\Longrightarrow \Uparrow g^{-1}(U) \subseteq g^{-1}(\Uparrow U)$

$\Longrightarrow \underbrace{\Uparrow g^{-1}(U)}_{\{0,1\}} \subseteq g^{-1}((\uparrow U)^\circ)$

$\Longrightarrow y = g(1) \in (\uparrow U)^\circ$.

E.G.

- smooth spacetimes
- interval topology of d.latt.
- (co)discrete spaces

Adjunction

$$\begin{tikzcd} \mathbf{OrdTop}_{oc} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc} \arrow[l, shift left] \end{tikzcd}$$

$$(S, \leqslant) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$$

$$(\mathrm{pt}(X), \boxed{?}) \longmapsfrom (X, \trianglelefteq)$$

Adjunction

OrdLoc $\longrightarrow$ **OrdTop**

$(X, \trianglelefteq) \longmapsto (pt(X), \leqslant)$

pt. ORD.

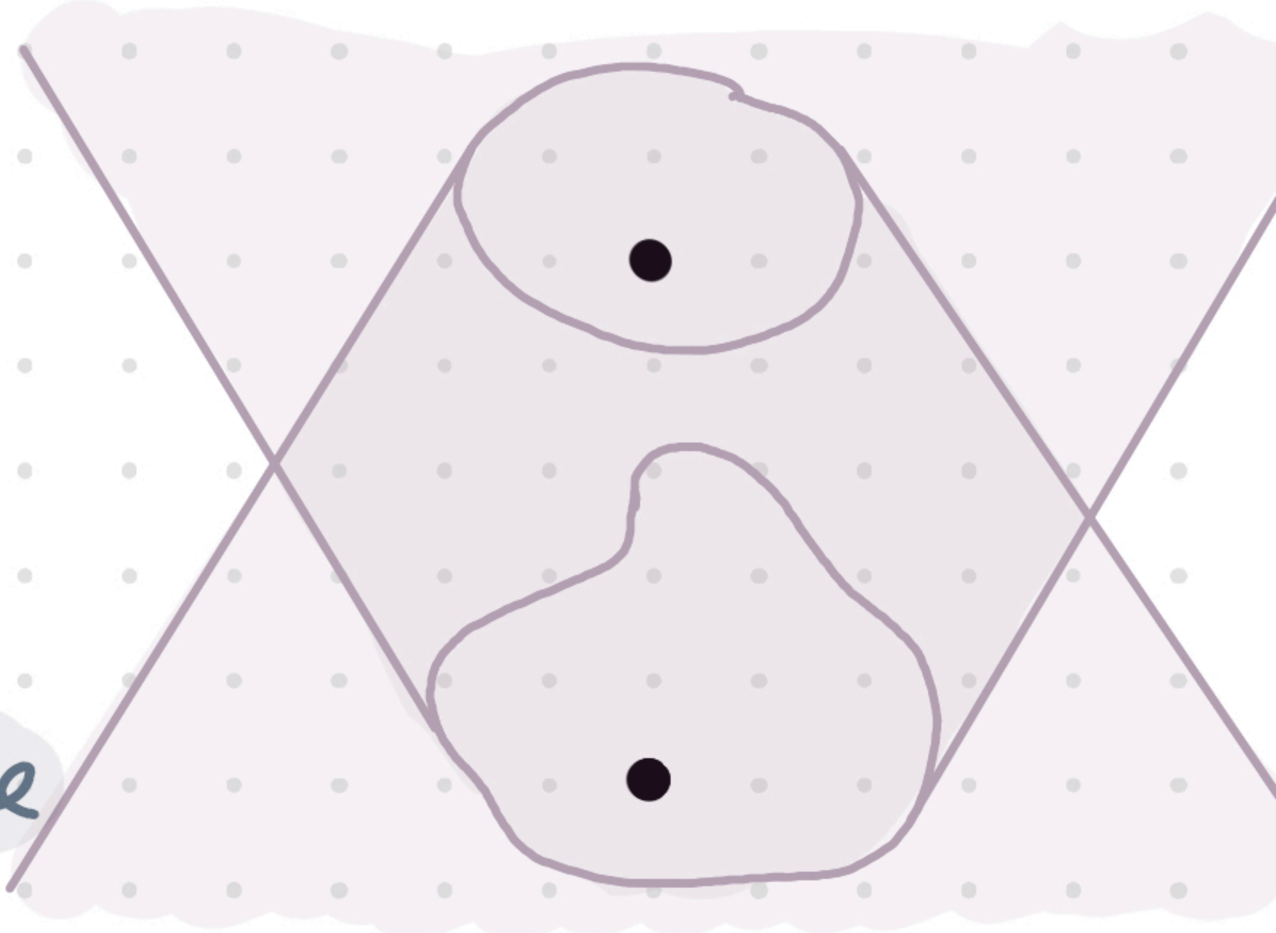
for $\mathcal{F}, \mathcal{G} \in pt(X)$:

$$\mathcal{F} \leqslant \mathcal{G} \iff \begin{array}{l} \forall U \in \mathcal{F} : \uparrow U \in \mathcal{G}, \\ \forall V \in \mathcal{G} : \downarrow V \in \mathcal{F}. \end{array}$$

$$x \leqslant y \iff \begin{array}{lll} \forall U \ni x & : & y \in \uparrow U, \\ \forall V \ni y & : & x \in \downarrow V. \end{array}$$

LEMMA

f monotone $\Rightarrow$ $pt(f)$ monotone



Adjunction

$$\mathbf{OrdLoc} \longrightarrow \mathbf{OrdTop}$$
$$(X, \trianglelefteq) \longmapsto (pt(X), \leqslant)$$

pt. ORD.

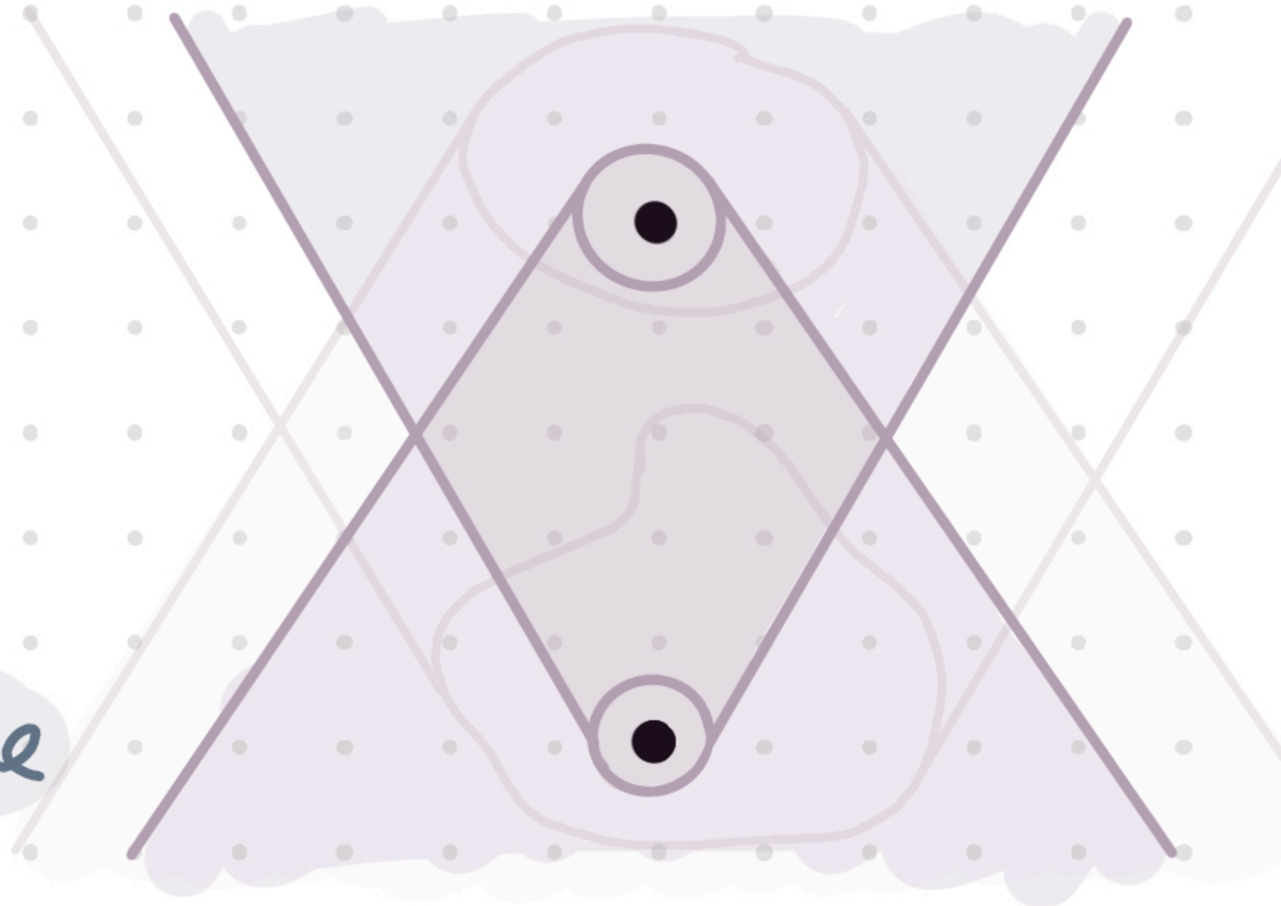
for $\mathcal{F}, \mathcal{G} \in pt(X)$:

$$\mathcal{F} \leqslant \mathcal{G} \iff \begin{array}{l} \forall U \in \mathcal{F} : \uparrow U \in \mathcal{G}, \\ \forall V \in \mathcal{G} : \downarrow V \in \mathcal{F}. \end{array}$$

$$x \leqslant y \iff \begin{array}{l} \forall U \ni x : y \in \uparrow U, \\ \forall V \ni y : x \in \downarrow V. \end{array}$$

LEMMA

f monotone $\Rightarrow$ $pt(f)$ monotone



Adjunction

$$\mathbf{OrdLoc} \longrightarrow \mathbf{OrdTop}$$
$$(X, \trianglelefteq) \longmapsto (\mathrm{pt}(X), \leqslant)$$

pt. ORD.

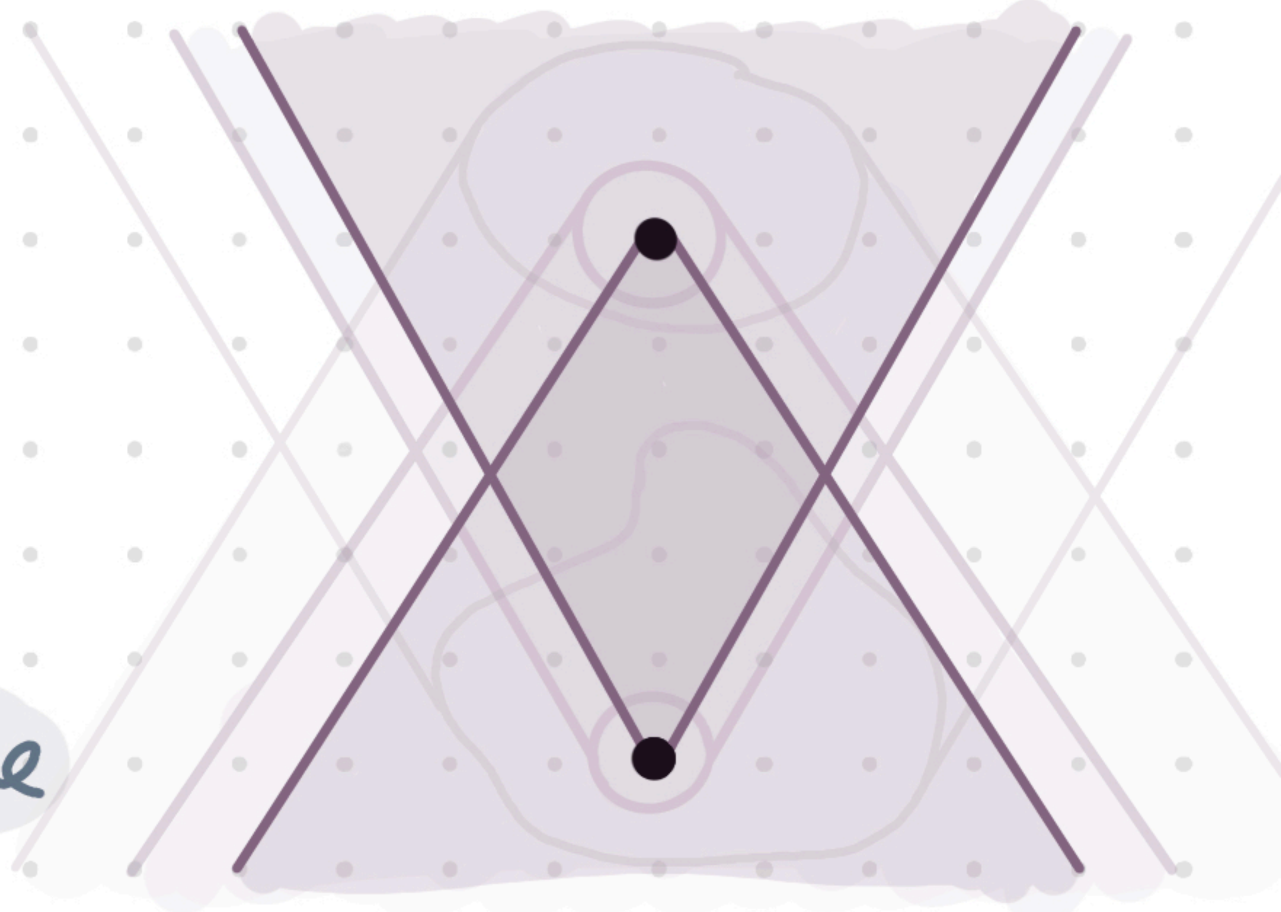
for $\mathcal{F}, \mathcal{G} \in \mathrm{pt}(X)$:

$$\mathcal{F} \leqslant \mathcal{G} \iff \begin{array}{l} \forall U \in \mathcal{F} : {\Uparrow} U \in \mathcal{G}, \\ \forall V \in \mathcal{G} : {\Downarrow} V \in \mathcal{F}. \end{array}$$

$$x \leqslant y \iff \begin{array}{l} \forall U \ni x \ : \ y \in {\uparrow} U, \\ \forall V \ni y \ : \ x \in {\downarrow} V. \end{array}$$

LEMMA

f monotone $\Rightarrow$ $\mathrm{pt}(f)$ monotone



Adjunction

$$\mathbf{OrdLoc} \longrightarrow \mathbf{OrdTop}_{oc}$$
$$(X, \trianglelefteq) \longmapsto (pt(X), \leqslant)$$

pt.ORD.

for $\mathcal{F}, \mathcal{G} \in pt(X)$:

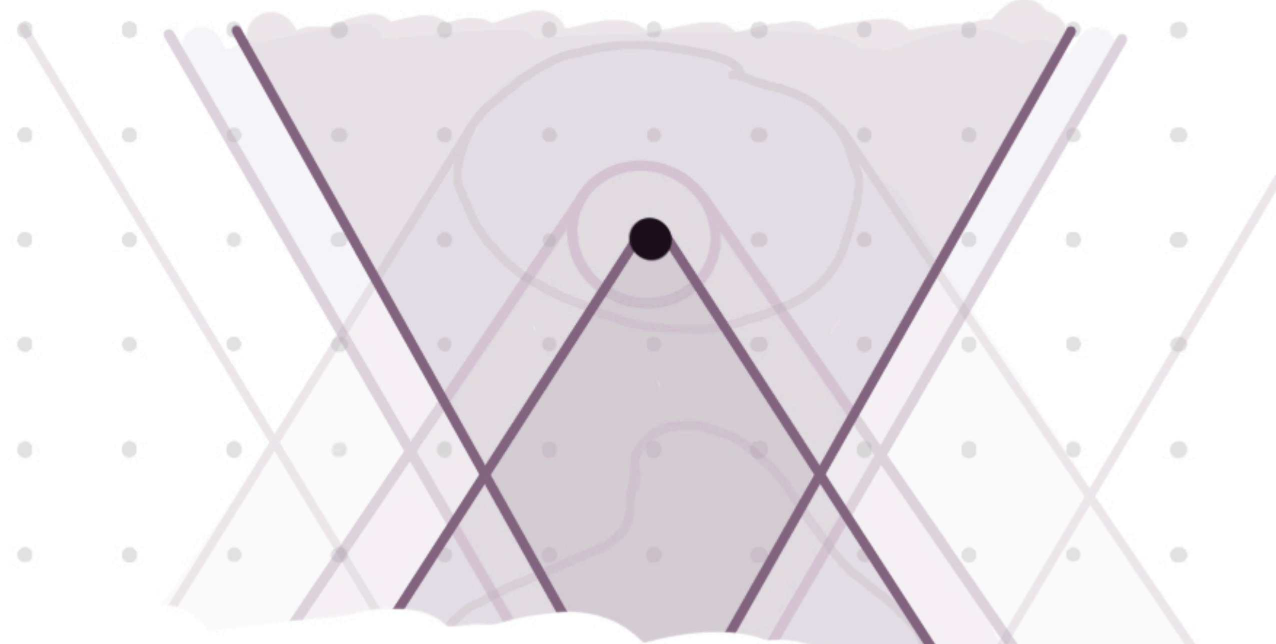
$$\mathcal{F} \leqslant \mathcal{G} \iff \begin{array}{l} \forall U \in \mathcal{F} : {\Uparrow} U \in \mathcal{G}, \\ \forall V \in \mathcal{G} : {\Downarrow} V \in \mathcal{F}. \end{array}$$

$$x \leqslant y \iff \begin{array}{l} \forall U \ni x : \; y \in {\uparrow} U, \\ \forall V \ni y : \; x \in {\downarrow} V. \end{array}$$

LEMMA

$pt(X)$ has OC if :

$$\uparrow pt(U) = pt({\Uparrow} U),$$
$$\downarrow pt(U) = pt({\Downarrow} U).$$



We define full subcat :

$$\mathbf{OrdLoc}^{\bullet}$$

Adjunction

$$
\begin{tikzcd}
\mathbf{OrdTop}^{\bullet}_{oc} \arrow[r, shift left=1.5, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc}^{\bullet} \arrow[l, shift left=1.5, "\mathrm{pt}"]
\end{tikzcd}
$$

$$(S, \leq) \longmapsto (\mathrm{Loc}(S), \trianglelefteq)$$

$$(\mathrm{pt}(X), \leq) \longmapsfrom (X, \trianglelefteq)$$

Adjunction

UNIT

$$\eta_S : \mathcal{S} \longrightarrow \mathrm{pt}\,\mathrm{Loc}(S)$$
$$x \longmapsto F_x := \{ U \in \mathcal{O}S : x \in U \}$$

LEMMA

η_S is monotone iff S has OC.

Adjunction

COUNIT

$$\varepsilon_X : \mathrm{Loc}\ \mathrm{pt}(X) \longrightarrow X$$

$$\varepsilon_X^{-1} : \mathcal{O}X \longrightarrow \mathcal{O}\mathrm{pt}(X)$$
$$U \longmapsto \mathrm{pt}(U)$$

LEMMA

ε_X is monotone $\iff \uparrow\mathrm{pt}(U) \subseteq \mathrm{pt}(\uparrow U)$

(even on **OrdLoc**)

Adjunction

THEOREM

$$
\begin{tikzcd}
\mathbf{OrdTop}_{oc}^{\bullet} \arrow[r, shift left=1.5, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc}^{\bullet} \arrow[l, shift left=1.5, "\mathrm{pt}"]
\end{tikzcd}
$$

Fixed Points

UNIT

$$\eta_S : S \longrightarrow \mathrm{pt}\,\mathrm{Loc}(S)$$
$$x \longmapsto \mathcal{F}_x := \{ U \in \mathcal{O}S : x \in U \}$$

LEMMA

η_S is iso. iff :

- S is sober ;
- S is T_0-ordered.

$$x \not\leq y \implies$$
$$\exists U \ni x : y \notin {\uparrow} U$$
or
$$\exists V \ni y : x \notin {\downarrow} V.$$

Fixed Points

COUNIT

$$\varepsilon_X : \mathrm{Loc}\,\mathrm{pt}(X) \longrightarrow X$$

$$\begin{aligned} \varepsilon_X^{-1} : \mathcal{O}X &\longrightarrow \mathcal{O}\mathrm{pt}(X) \\ U &\longmapsto \mathrm{pt}(U) \end{aligned}$$

LEMMA

ε_X is iso iff X is spatial

THEOREM

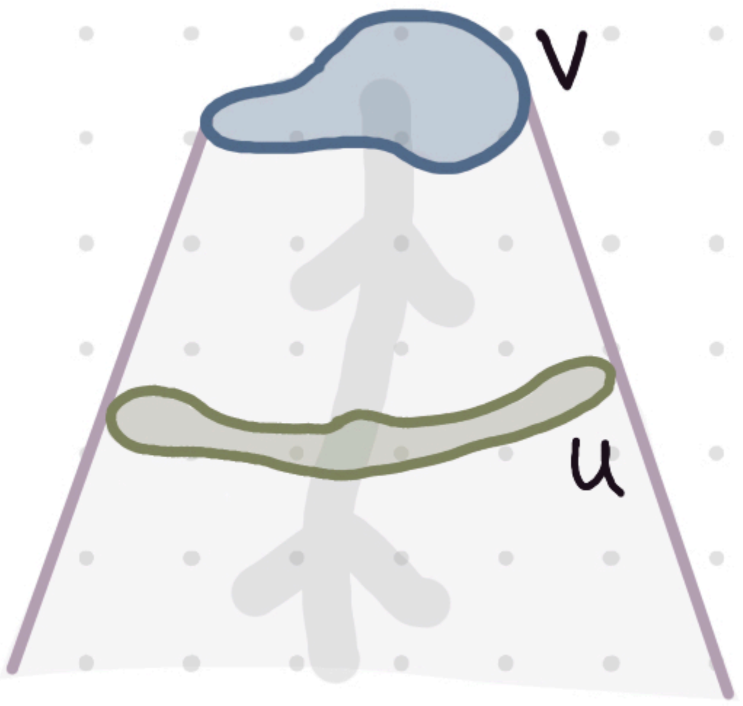
$$\begin{tikzcd} \mathbf{OrdTop}_{OC}^{\bullet} \arrow[r, shift left=1.5, "\mathrm{Loc}"] \arrow[r, phantom, "\perp"] & \mathbf{OrdLoc}^{\bullet} \arrow[l, shift left=1.5, "\mathrm{pt}"] \end{tikzcd}$$

COROLLARY

$$\left\{\begin{array}{c}\text{sober } T_0\text{-ordered} \\ \text{spaces with OC}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{spatial ordered} \\ \text{locales with } \bullet\end{array}\right\}$$

Current Work

causal coverages



all info. reaching V <u>must</u> pass through U

Grothendieck topology on $\mathcal{K}\ell(\downarrow)$

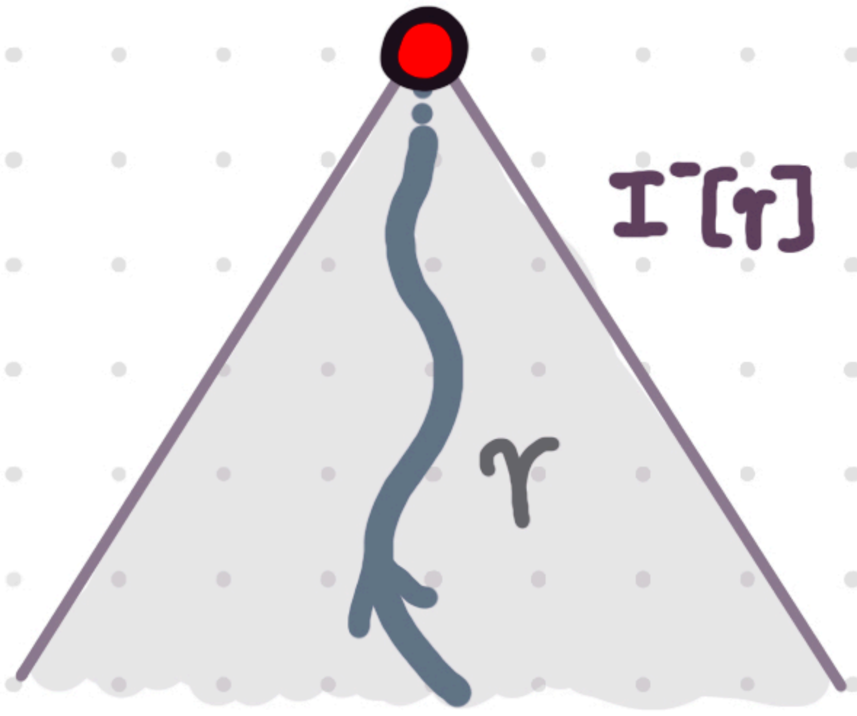
Abstract Domains of Dependence

Current Work

Causal boundary

LEMMA ideal points correspond to primes in $\mathrm{im}(\uparrow)$.

($p \neq 1$, $p = x \wedge y$ implies $p = x$ or y)



[Geroch-Kronheimer-Penrose 72, Thm. 2.1]