AXIOMS FOR THE CATEGORY OF: HILBERT SPACES & LINEAR CONTRACTIONS

arXiv:2211.02688

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EDINBURGH CATEGORY THEORY SEMINAR
18 JANUARY 2023

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- MAIN CONSTRUCTION
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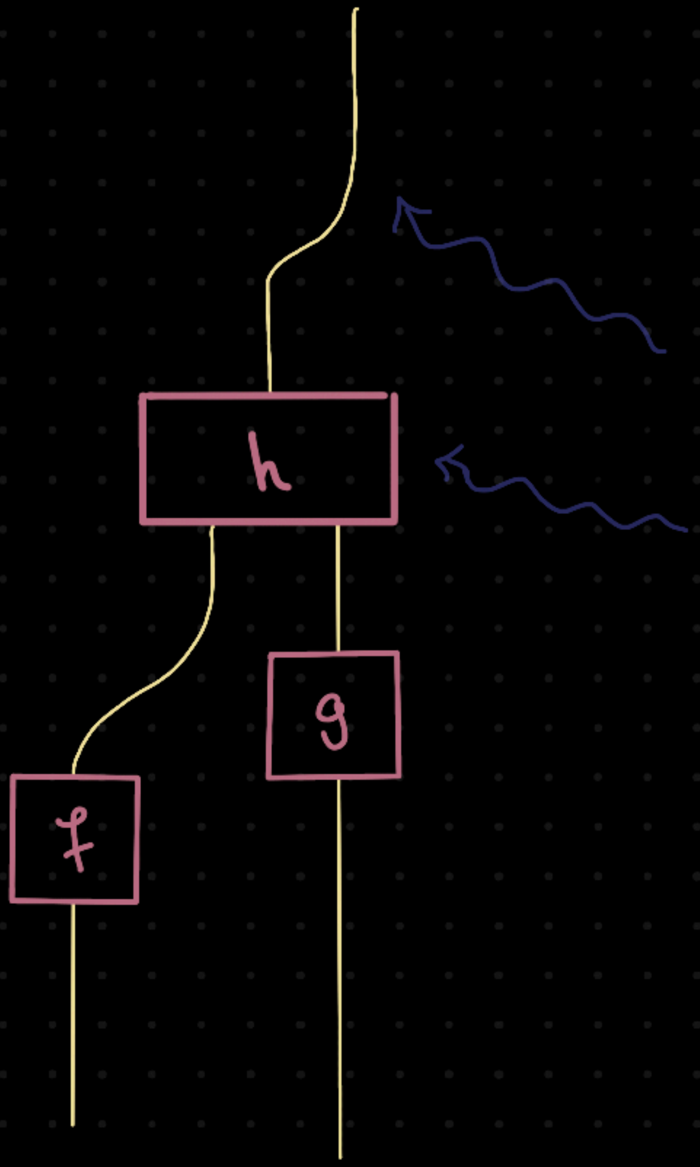
PHYSICS AS PROCESSES.

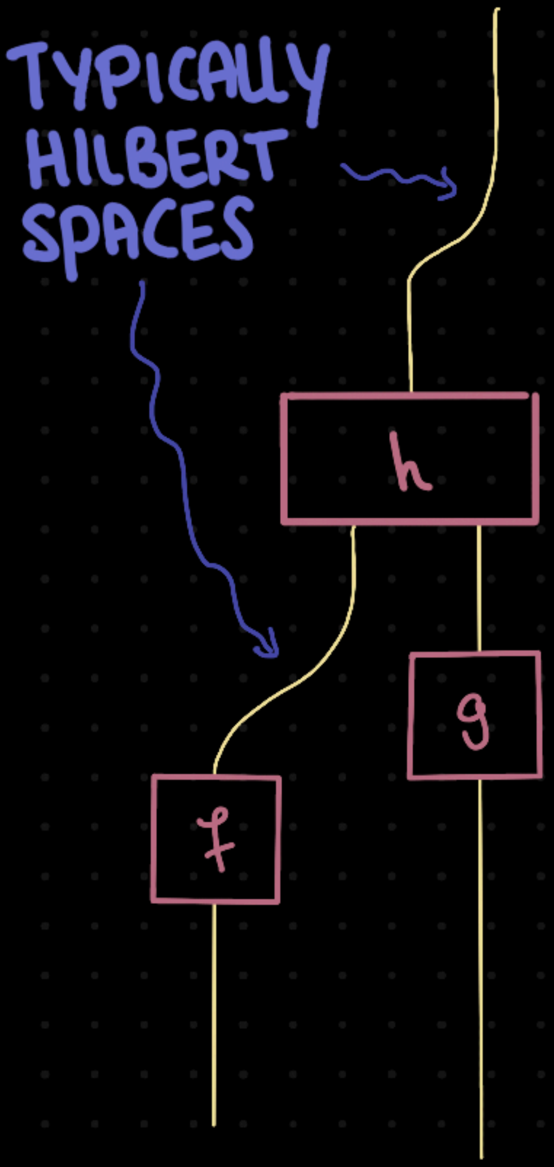


MONOIDAL CATEGORIES:

OBJECTS = PHYSICAL SYSTEMS
ARROWS = PHYSICAL PROCESSES

COMPOSITION = SEQUENTIAL COMPOSITION
TENSOR = PARALLEL COMPOSITION

PHYSICS AS PROCESSES.

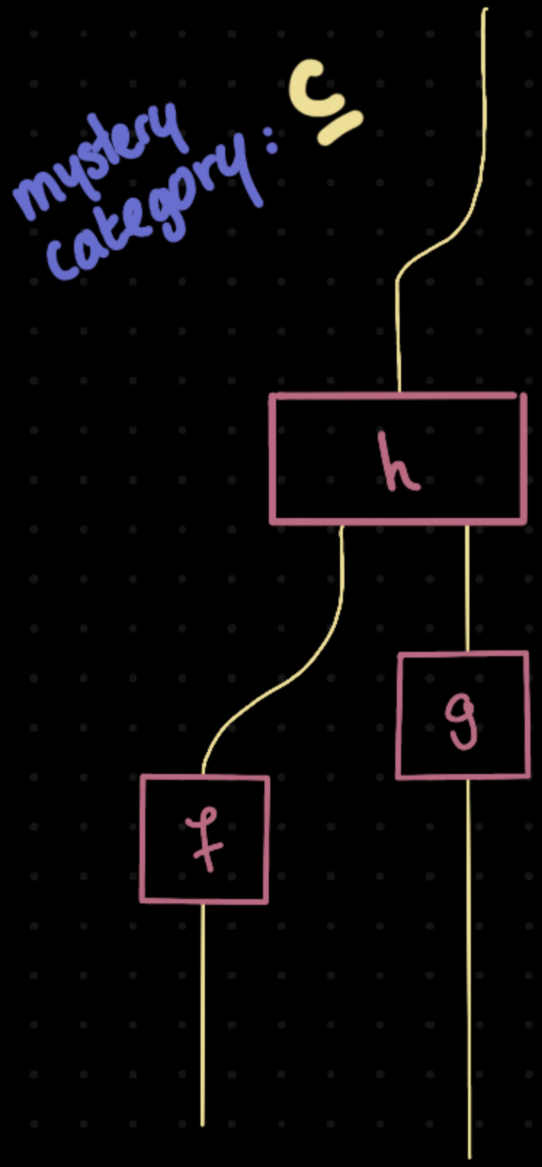


CATEGORICAL QUANTUM MECHANICS: **QM**

HILB
- objects : HILBERT SPACES
- arrows : BOUNDED LINEAR MAPS

CON
- objects : IDEM.
- arrows : LINEAR f s.t.: $\|f\| \leq 1$

POSING THE QUESTION.



HOW CAN WE TELL OUR PROCESSES ARE QUANTUM ?

WHAT ARE PROPERTIES OF $\underline{C}$ SUCH THAT:

$\underline{C} \simeq \underline{QM}$?

(AS DAGGER MONOIDAL CATS.)

ANSWERS?

- HILB: HEUNEN + KORNELL, PNAS 2022 Vol. 119 No. 9, arXiv: 2109.07418.

THIS TALK

- CON: HEUNEN + KORNELL + vdS, arXiv: 2211.02688.

- ONGOING: HILB_A, $\underbrace{\text{FHILB}, \text{UNITARY},}_{\text{HEUNEN + DIMEGLIO}}$ ...
  (NOT BY ME!!)

RECAP OF THE OLD RESULT.

HEUNEN + KORNELL
PNAS 2022 Vol. 119 No. 9
arXiv: 2109.07418.

THE AXIOMS:

(A) $\mathcal{C}$ IS DAGGER MONOIDAL

$\dagger : \mathcal{C}^{op} \longrightarrow \mathcal{C}$, id.-on-obj., idempotent, $id_H^\dagger = id_H$.

Monoidal str. whose coherence morphisms are $\dagger$-iso., $\varphi^{-1} = \varphi^\dagger$.

(B) I IS A SIMPLE SEPARATOR

SIMPLE: I has two subobjects.

MON. SEPARATOR: if $\forall I \xrightarrow{h} H \; \forall I \xrightarrow{k} K$: $f \circ (h \otimes k) = g \circ (h \otimes k)$, then $f = g$.

(C) $\mathcal{C}$ HAS $\dagger$-BIPRODUCTS

$\mathcal{C}$ has a zero obj. 0. Coproducts:

$$\begin{tikzcd} H \arrow[dr, "i"'] & & K \arrow[dl, "j"] \\ & H \oplus K & \end{tikzcd}$$

$\dagger$-monomorphisms ← $f^\dagger \circ f = id.$

$j^\dagger \circ i = 0$

(D) $\mathcal{C}$ HAS $\dagger$-EQUALISERS

All equalisers exist, and they are $\dagger$-monomorphisms

(E) $\dagger$-MONOS ARE $\dagger$-KERNELS

Any $\dagger$-mono f is a $\dagger$-kernel, i.e. an equaliser:

$$N \xrightarrow{f} K \underset{0}{\overset{\exists}{\rightrightarrows}} H$$

(F) $\mathcal{C}$ HAS DIRECTED COLIMITS OF $\dagger$-MONOS

RECAP OF THE OLD RESULT.

HEUNEN + KORNELL
PNAS 2022 Vol. 119 No. 9
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THEOREM. IF $\mathbf{C}$ SATISFIES AXIOMS (A) — (F), THEN:

$$\mathbf{C} \simeq \mathbf{HILB}.$$

(AS DAGGER MONOIDAL CATS.)

THE STRATEGY.

mystery category: $\underline{D}$

Axioms $\underline{\text{HILB}}$ (KNOWN) $\longrightarrow$ Axioms $\underline{\text{CON}}$

SO WE NEED TO FIGURE OUT:

HOW ARE $\underline{\text{HILB}}$ AND $\underline{\text{CON}}$ RELATED?

THE IDEA:

$$\underline{\mathrm{HILB}}(H,K)\setminus\{0\} \leftrightsquigarrow \underline{\mathrm{CON}}(H,K)\setminus\{0\}$$

$$f \longmapsto f/\|f\|$$

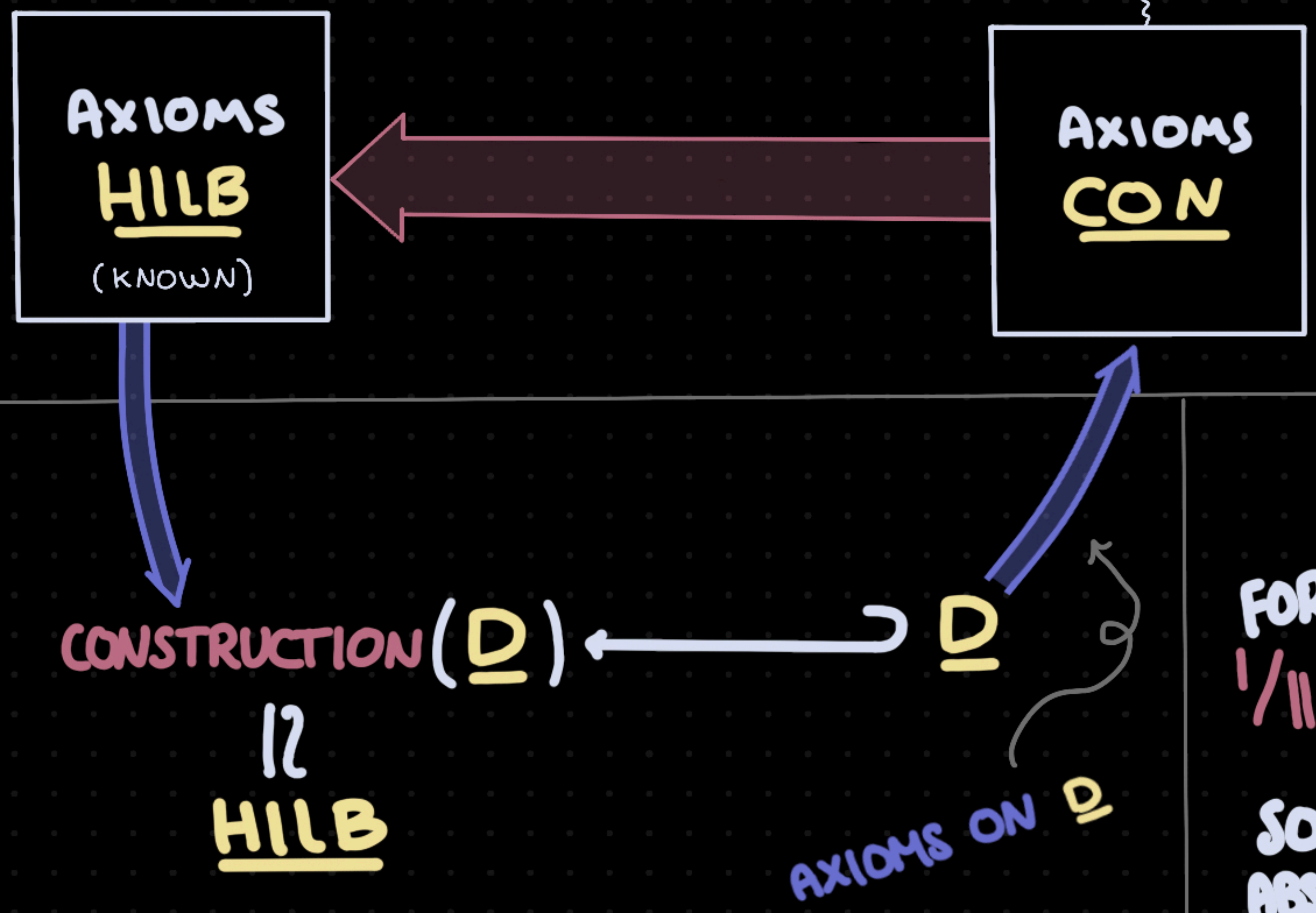
$$f/\|f\| \longmapsfrom f$$

THE PROBLEM:

FOR THE "$\longleftarrow$" DIRECTION, $1/\|f\|$ DOESN'T "EXIST" IN $\underline{\text{CON}}$.

SO HOW DO WE DO THIS ABSTRACTLY IN mystery category $\underline{D}$?

THE STRATEGY.



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HOW ARE **HILB** AND **CON** RELATED?

THE PROBLEM:

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SO HOW DO WE DO THIS ABSTRACTLY IN *mystery category* $\underline{C}$?

SCALARS.

IN A MONCAT. $\underline{C}$ THERE ARE **SCALARS**:

$$z : I \longrightarrow I$$

THEY FORM A **COMMUTATIVE MONOID** UNDER COMPOSITION

SCALARS z AND MORPHISMS f CAN BE MULTIPLIED:

$$\begin{tikzcd}
H \arrow[rr, "z \bullet f"] \arrow[d, "\wr"] & & K \\
I \otimes H \arrow[rr, "z \otimes f"'] & & I \otimes K \arrow[u, "\wr"']
\end{tikzcd}$$

FROM FUNCTORIALITY OF $\otimes$ WE GET:

$$z \bullet (g \circ f)$$
$$\|$$
$$(z \bullet g) \circ f \qquad \text{“SHUFFLING”}$$
$$\|$$
$$g \circ (z \bullet f)$$

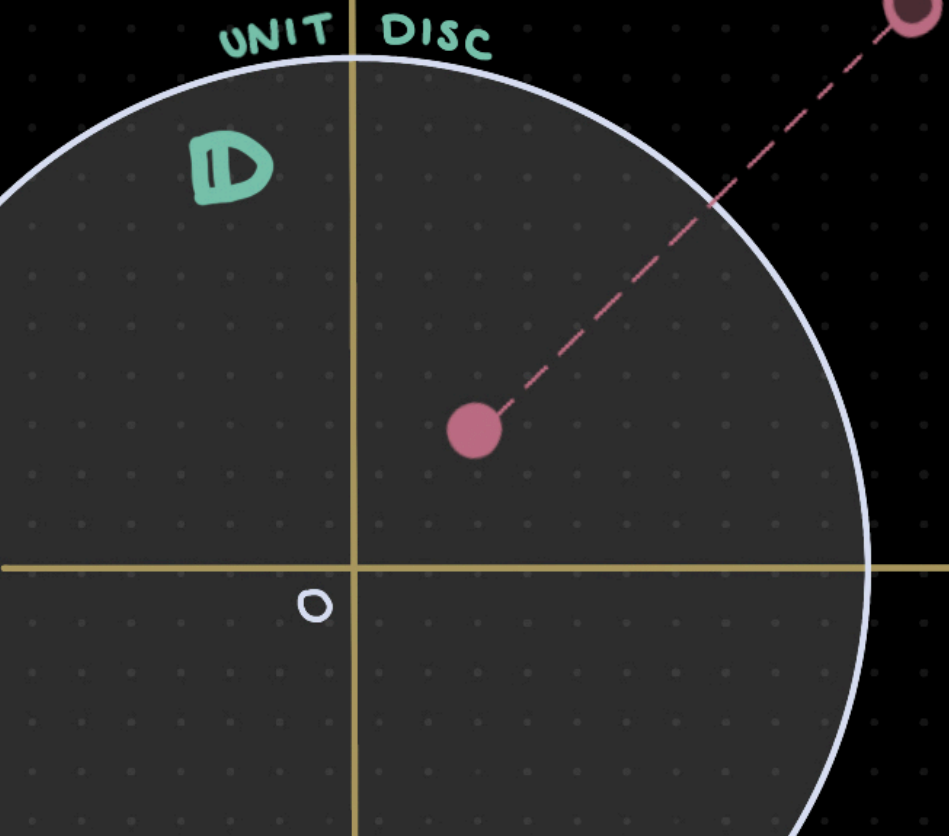
SCALARS IN HILB & CON.

$\mathbb{C}$

complex numbers

WE FIND:

$\underline{\text{HILB}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}$ AND $\underline{\text{CON}}(\mathbb{C},\mathbb{C}) \cong \mathbb{D}$.



THE IDEA: CONSTRUCTION(—) ADDS FORMAL INVERSES FOR ALL SCALARS.

THEN: CONSTRUCTION($\underline{\text{CON}}$) $\cong$ $\underline{\text{HILB}}$.

ROUGHLY: id.-on-obj., and send formal inverses to ACTUAL inverse in HILB

THE NEW AXIOMS.

(1) $\underline{D}$ IS A $\dagger$- CAT.

(2) $\underline{D}$ IS A $\dagger$- RIG CAT.

Two $\dagger$-monoidal structures:
$(\otimes, I)$ and $(\oplus, 0)$ such that
$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$
and $\otimes$ distributes over $\oplus$.

ALMOST BIPRODUCTS

(3) $(\oplus, 0)$ IS AFFINE

0 is initial, and hence a zero obj.
This gives natural:

$inl_{HK} : H \longrightarrow H \oplus K$

$inr_{HK} : K \longrightarrow H \oplus K$

(4) inl, inr ARE JOINTLY EPIC

$$\left.\begin{array}{l} f \circ inl = g \circ inl \\ f \circ inr = g \circ inr \end{array}\right\} \Longrightarrow f = g$$

(5) THERE IS MIXTURE

$\exists\, s : I \longrightarrow I \oplus I$ with
$inl^\dagger \circ s \neq 0 \neq inr^\dagger \circ s$

UNIT AXIOMS

(6) I IS $\dagger$- SIMPLE

(7) I IS A $\otimes$-SEPARATOR

$\dagger$-AXIOMS

(8) $\underline{D}$ HAS ALL $\dagger$- EQUALISERS

(9) $\dagger$- MONOS ARE $\dagger$- KERNELS

(10) SUBOBJECTS ARE DETERMINED BY POSITIVE MAPS

$s = t$ as subobj. iff $s \circ s^\dagger = t \circ t^\dagger$

(11) $\underline{D}$ HAS ALL DIRECTED COLIMITS

TOWARDS THE CONSTRUCTION.

SOME LEMMAS :

($\dagger$-AXIOMS)

LEMMA. MORPHISMS FACTOR AS

$$H \xrightarrow{\text{epi}} E \xrightarrow{\dagger\text{-mono}} K$$

(UNIT AXIOMS)

LEMMA. NON-ZERO SCALARS ARE MONIC & EPIC

$0 \neq z : I \longrightarrow E \xrightarrow{m} I$,
by (6) we get m iso.

LEMMA. IF $z \neq 0$, THEN $z \cdot f = z \cdot g$ IMPLIES $f = g$

THE CONSTRUCTION.

GIVEN $\begin{cases} \text{MON CAT}: \underline{D} \\ \text{SCALARS}: \mathbb{D} := \underline{D}(I, I) \end{cases}$

THERE IS A CAT.

$$\text{CONSTRUCTION}(\underline{D}) = \underline{D}(\mathbb{D}^{-1})$$

WITH:

OBJECTS: SAME AS $\underline{D}$

ARROWS: OF THE FORM

$$H \xrightarrow{[t/z]} K$$

WHERE $t \in \underline{D}(H, K)$

$z \in \mathbb{D} \setminus \{0\}$

UNDER EQUIVALENCE RELATION:

$$(t/z) \sim (t'/z') \iff z' \bullet t = z \bullet t'$$

(TRANSITIVE BY LEMMA.)

IDENTITY: $[id_H/1]$

COMPOSITION:

$$[t/z] \circ [s/w] := [t \circ s / z \bullet w]$$

(WELL DEFINED BY SHUFFLING & MONIC SCALARS.)

THE CONSTRUCTION.

THE CONSTRUCTION IS A LOCASISATION:

WE HAVE A FAITHFUL INCLUSION FUNCTOR:

$$\begin{array}{ccc} \underline{D} & \hookrightarrow & \underline{D}(\mathbb{D}^{-1}) \\ f & \longmapsto & [f/1] \end{array}$$

PROPOSITION. IF F IS STRONG $\otimes$-MONOIDAL AND $\forall z \in \mathbb{D} \setminus \{0\} : F(z)$ INVERTIBLE:

$$\begin{tikzcd} \underline{D} \arrow[r, hook] \arrow[dr, "F"'] & \underline{D}(\mathbb{D}^{-1}) \arrow[d, dashed, "\exists! \text{ STRONG } \otimes\text{-MONOIDAL}"] \\ & \underline{C} \end{tikzcd}$$

THE CONSTRUCTION.

AFTER A LOT OF DETAILS... :

THEOREM. IF $\underline{D}$ SATISFIES AXIOMS (I) — (II) THEN $\underline{D}(\mathbb{D}^{-1})$ SATISFIES AXIOMS (A) – (F), SO:

$$\underline{D} \hookrightarrow \underline{D}(\mathbb{D}^{-1}) \cong \underline{\mathrm{HILB}}$$

TOWARDS THE THEOREM.

HENCE $\underline{D} \subseteq \underline{\mathrm{HILB}}$.
WE ARE "JUST" LEFT TO SHOW:
$\underline{D} = \underline{\mathrm{CON}}$.

$\underline{D}(\mathbb{D}^{-1})$ HAS DAGGER $[f/z]^\dagger := [f^\dagger/z^\dagger]$.

LEMMA. SUPPOSE $t^\dagger \circ t = z^\dagger \bullet z \bullet id$, FOR $z \in \mathbb{D}$. THEN:
$\exists\, \dagger$-MONO $m : t = z \bullet m$.

proof. $t : \bullet \xrightarrow{e} \bullet \rightarrowtail^{m} \bullet$

THEN $e^\dagger \circ e = (z \bullet id)^\dagger \circ (z \bullet id)$
SO (10) GIVES
$\exists\, \dagger$-iso. $u : e = z \bullet u$

POSITIVE MAPS DETERMINE SUBOBJECTS

AND HENCE:
$$t = z \bullet \underbrace{(m \circ u)}_{\dagger\text{-mono.}}.$$

PROPOSITION. THERE IS AN ISO.:
$$\underline{D}_{\dagger m} \xhookrightarrow{\sim} \underline{D}(\mathbb{D}^{-1})_{\dagger m}$$

$\dagger$-monics

proof. WE JUST NEED TO SHOW THE INCLUSION FUNCTOR IS FULL. SO LET $[t/z]$ BE $\dagger$-MONO.
THIS MEANS:
$$t^\dagger \circ t = z^\dagger \bullet z \bullet id.$$
HENCE $t = z \bullet m$ AND
$$m \longmapsto [m/1] = [t/z].$$

TOWARDS THE THEOREM.

HENCE $\underline{D} \subseteq \underline{HILB}$.
WE ARE "JUST" LEFT TO SHOW:
$\underline{D} = \underline{CON}$.

PROPOSITION. $\mathbb{D} = \{ z \in \underset{\text{actual complex numbers}}{\mathbb{C}} : |z| \leq 1 \}$.

Proof. FOR FINITE $R \subseteq \mathbb{N}$ DEFINE:

$$I^R := \underbrace{I \oplus \cdots \oplus I}_{|R|\text{-times}}.$$

WE GET COLIMIT IN $\underline{HILB}$:

$$\begin{tikzcd} \cdots I^R \arrow[rr, tail, "i_{RS}", "R \subseteq S"'] \arrow[dr, bend right, "i_R"'] & & I^S \cdots \arrow[dl, bend left, "i_S"] \\ & I^{\mathbb{N}} & \end{tikzcd}$$

$\dagger$-MONOS ARE THE SAME IN $\underline{HILB}$ & $\underline{D}$.

SO WE GET COLIMIT IN $\underline{D}$: $j_R : I^R \rightarrowtail H$.

FOR $z \in \mathbb{D}$, CONSTRUCT NATURAL

$$t_R : I^R \longrightarrow I^R,$$
$$t_{\{n\}} : I \longrightarrow I$$
$$s \longmapsto z^n \cdot s$$

$$\begin{tikzcd} I \arrow[r, tail, "j_{\{n\}}"] \arrow[d, "t_{\{n\}}"'] & H \arrow[d, "t"] \\ I \arrow[r, tail, "j_{\{n\}}"'] & H \end{tikzcd}$$

HENCE t HAS EIGENVALUE z^n, $\forall n$. SO:

$$\|t\| \geq |z|^n.$$

SINCE t IS BOUNDED:

$$|z| \leq 1.$$

TOWARDS THE THEOREM.

HENCE $\underline{\mathrm{D}} \subseteq \underline{\mathrm{HILB}}$.
WE ARE "JUST" LEFT TO SHOW:
$\underline{\mathrm{D}} = \underline{\mathrm{CON}}$.

LEMMA. WE HAVE: $\underline{\mathrm{D}}(H,K) \subseteq \{ t \in \underline{\mathrm{HILB}}(H,K) : \|t\| \leq 1 \}$.

proof. LET $t \in \underline{\mathrm{D}}(H,K)$. SFTSOC $\|t\| > 1$.

THEN $\exists$ UNIT VECTOR $x : I \longrightarrow H$, $\|t \circ x\| > 1$.

x IS $\dagger$-MONO, SO $x \in \underline{\mathrm{D}}(I,H)$. BUT THEN: $\lightning$

$$\mathbb{D} \ni x^\dagger \circ t^\dagger \circ t \circ x = \|t \circ x\|^2 > 1.$$

CONCLUSION: $\underline{\mathrm{D}} \hookrightarrow \underline{\mathrm{D}}(\mathbb{D}^{-1}) \simeq \underline{\mathrm{HILB}}$ LANDS IN $\underline{\mathrm{CON}}$.

TOWARDS THE THEOREM.

HENCE $\mathbf{D} \subseteq \mathbf{HILB}$.
WE ARE "JUST" LEFT TO SHOW:
$\mathbf{D} = \mathbf{CON}$.

LEMMA. WE HAVE: $\mathbf{D}(H,K) \supseteq \{ t \in \mathbf{HILB}(H,K) : \|t\| \leq 1 \}$.

proof. FIRST: $t : H \longrightarrow H$, $\|t\| < 1$. BY RUSSO-DYE-GARDNER:

$$t = \frac{1}{n}\left(u_1 + \cdots + u_n \right)$$

†-isomorphisms in HILB

THE NORMALISED DIAGONAL: $\omega : H \longrightarrow H \oplus \cdots \oplus H$, $x \longmapsto (x, \cdots x)/\sqrt{n}$
IS †-MONO IN HILB, AND:

$$t = \underbrace{\omega^\dagger \circ \left(u_1 \oplus \cdots \oplus u_n \right) \circ \omega}_{\text{IN } \mathbf{D}}$$

IN GENERAL: POLAR DECOMPOSITION IN HILB:

$$t = \underbrace{u \circ v^\dagger}_{\dagger\text{-monos}} \circ \underbrace{s}_{\text{previous case}}$$

TOWARDS THE THEOREM.

HENCE $\underline{\mathbf{D}} \subseteq \underline{\mathbf{HILB}}$.
WE ARE "JUST" LEFT TO SHOW:
$$\underline{\mathbf{D}} = \underline{\mathbf{CON}}.$$

LEMMA. WE HAVE: $\underline{\mathbf{D}}(H,K) \supseteq \{ t \in \underline{\mathbf{HILB}}(H,K) : \|t\| \leq 1 \}$.

proof. LASTLY: $\|t\| = 1$. TAKE

$0 < z_1 < z_2 < \cdots < 1, \quad \sup z_n = 1.$

THEN $\|z_n \bullet t\| < 1$, SO $z_n \bullet t \in \underline{\mathbf{D}}(H,K)$.

IN $\underline{\mathbf{D}}$:

COLIMIT IN $\underline{\mathbf{D}}$

$$\begin{tikzcd}
H \arrow[r, "z_1/z_2"] \arrow[drrr, bend right, "z_1"] \arrow[ddrrr, bend right, "z_1 \bullet t"'] & H \arrow[r, "z_2/z_3"] \arrow[drr, bend right, "z_2"] \arrow[ddrr, bend right, "z_2 \bullet t"'] & H \arrow[r] \arrow[dr, "z_3"] \arrow[ddr, "z_3 \bullet t"'] & \cdots \\
 & & & H \arrow[d, "\exists! S"] \\
 & & & K
\end{tikzcd}$$

HENCE WE GET IN PARTICULAR:

$$\begin{aligned}
& z_1 \bullet S \\
& \quad \| \quad (\text{SHUFFLING}) \\
& S \circ (z_1 \bullet \mathrm{id}) \\
& \quad \| \quad (\text{DIAGRAM}) \\
& z_1 \bullet t
\end{aligned}$$

AND SINCE $z_1 \neq 0$:

$$t = S \in \underline{\mathbf{D}}(H,K).$$

THE THEOREM.

THEOREM. IF $\underline{D}$ SATISFIES AXIOMS (I) — (III), THEN:

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(AS DAGGER RIG CATS.)

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Thanks very much for your attention! :)