A Note on

Morita Equivalence and C*-correspondences

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Abstract

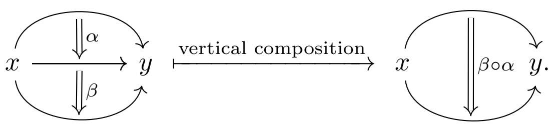
In [**Rie74b**], Rieffel introduced *(strong) Morita equivalence* of C*-algebras. Two C*-algebras are said to be Morita equivalent whenever there exists a so-called *equivalence bimodule* between them. On the other hand, we have the more general notion of a *C*-correspondence*. These generalise ∗-homomorphisms, and form the arrows of a bicategory **C*Corr**. Two C*-algebras are *equivalent* in this bicategory if there exists a *weakly invertible* C*-correspondence between them. In this note we provide a succinct account of what these notions mean, and prove that the two notions of (Morita) equivalence coincide.

1 The idea of a bicategory

The notion of Morita equivalence was first introduced by Morita in [**Mor58**] to study modules over rings. Vaguely speaking, two rings are *Morita equivalent* if their representation theory is equivalent. This definition was later extended to the setting of C*-algebras by Rieffel, first appearing in [**Rie74b**].

Almost a decade after Morita's publication, we saw the introduction of *bicategories* by Bénabou in [**Bén67**], and it became apparent that this was the right setting to develop Morita's theory. (See [**JF15**], and references therein, for more historical context.) We shall outline below, quite informally, what it means for two objects in a bicategory to be equivalent. Then, in Section **2**, we shall apply this framework to the setting of C*-algebras, and we will show that the bicategorical equivalences are exactly the equivalence bimodules as introduced by Rieffel.

The notion of a *2-category* arises quite immediately in the study of categories. It is an elementary fact that the category **Cat** of all (small) categories with functors and natural transformations is a 2-category[1]. Besides arrows between objects (as is inherent to any category), a 2-category contains also morphisms between those arrows. These are called *2-morphisms*, or *2-arrows*. For **Cat** they are the natural transformations between functors. They behave nicely with respect to their composition in the sense that for any two objects x and y in a 2-category $\mathbf{C}$, their set of 1-morphisms $\mathrm{Hom}_{\mathbf{C}}(x, y)$ form a genuine category when taking the 2-morphisms of $\mathbf{C}$ as their arrows. In this way a 2-category can also be seen as a family of categories, indexed by pairs of objects in $\mathbf{C}$. The composition in $\mathrm{Hom}_{\mathbf{C}}(x, y)$ of 2-morphisms is called *vertical composition*. This may be depicted diagrammatically as



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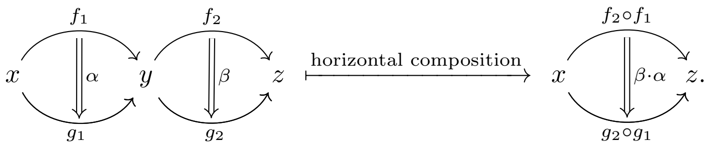
[1] "**Cat** is the mother of all 2-categories, just as **Set** is the mother of all categories," [**Lac07**].

There is also a *horizontal composition*, commuting with vertical composition, in the guise of a functor $\mathrm{Hom}_{\mathbf{C}}(x,y) \times \mathrm{Hom}_{\mathbf{C}}(y,z) \to \mathrm{Hom}_{\mathbf{C}}(x,z)$:



The additional structure allows us to think of isomorphism between arrows, called *2-isomorphism*, which generalises the notion of strict equality between objects. This leads to, among other things, the notion of a *2-commutative* (or *weakly commutative*) diagram, which is a diagram that commutes only up to 2-isomorphism. For example, we would say the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

2-commutes if there exists a 2-isomorphism $\alpha : g \circ f \Rightarrow k \circ h$, in which case we also write $g \circ f \cong k \circ h$. The notion of commuting only up to 2-isomorphism occurs already very early on in the theory of categories. Recall that a functor $F : \mathbf{C} \to \mathbf{D}$ is called an *equivalence of categories* if there exists another functor $G : \mathbf{D} \to \mathbf{C}$ such that there are natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathbf{D}}$ and $G \circ F \cong \mathrm{id}_{\mathbf{C}}$. In other words, F is invertible *up to 2-isomorphism*.

For our (and many other) purposes, the notion of 2-category is too restrictive. (The 2-categories we describe are often called *strict 2-categories*.) What we need instead are *weak 2-categories*, also known as *bicategories*. Intuitively we may think of a bicategory as a category where every axiom holds merely up to 2-isomorphism. In particular, a bicategory is a 2-category where composition is not strictly associative, nor unital. Therefore, in a bicategory, whenever we have three composable arrows $f : x \to y$, $g : y \to z$ and $h : z \to w$, say, we can only hope to have 2-isomorphisms

$$(f \circ g) \circ h \Longrightarrow f \circ (g \circ h), \qquad f \circ \mathrm{id}_x \Longrightarrow f, \qquad \text{and} \qquad \mathrm{id}_y \circ f \Longrightarrow f,$$

instead of full-fledged equalities. These canonical 2-isomorphisms are subject to various *coherence axioms*. We omit them here. For the precise definition of a bicategory and more details, we refer to [**Mac98**]. Also see [**Lac07**] for a modern overview. It should be noted that a strict 2-category is a special case of a bicategory, where the three 2-isomorphisms in the previous equation are always just the identity maps.

In a bicategory there are three degrees of sameness for objects. The strictest form is simply equality: $x = y$. Then there is the familiar notion of *isomorphism*: $x \cong y$, which means there are two arrows $f : x \to y$ and $f^{-1} : y \to x$ satisfying $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$. The map f is then known as a *strict 1-isomorphism*, or just as an *isomorphism*. These two concepts make sense in any category, but in a bicategory we have an additional notion:

Definition 1.1. Let $\mathbf{C}$ be a bicategory, and consider two objects $x, y \in \mathrm{ob}(\mathbf{C})$. We say x and y are *equivalent* (or *weakly isomorphic*) if there exists an arrow $f : x \to y$ that is invertible up to 2-isomorphism. In other words, if there are arrows $f : x \to y$ and $g : y \to x$ satisfying $f \circ g \cong \mathrm{id}_y$ and $g \circ f \cong \mathrm{id}_x$.

Note that this generalises the notion of equivalence between categories to the objects of an arbitrary bicategory. We leave it to the reader to fill in the gaps on why this forms a genuine equivalence relation on the set of objects. To distinguish equivalences that occur inside of a bicategory from other notions of equivalence (such as from an equivalence relation), we might call them *bicategorical* equivalences.

2 C*-algebras and C*-correspondences

Our first point of order will be to give a precise construction of the bicategory **C*Corr** of C*-correspondences between C*-algebras (Theorem **2.14**). For that, we will provide a brief recollection of the definition of a Hilbert C*-module in Section **2.1**. In light of Definition **1.1** there follows a notion of equivalence between C*-algebras that is inherent to **C*Corr**. In order to justify the claim that bicategories form the correct setting for Morita theory in the C*-algebra setting, we will prove (Theorem **2.27**) that the equivalences in **C*Corr** are exactly the equivalence bibundles (Definition **2.16**) inducing Morita equivalences. In this sense we will have justified that these two natural definitions of (Morita) equivalence are, in fact, equivalent.

Recall that a *C*-algebra* A is a Banach $*$-algebra satisfying the *C*-identity*: for all $a \in A$ we have $\|a^*a\| = \|a\|^2$. We do not assume that our C*-algebras are unital. Morphisms between C*-algebras are $*$-homomorphisms, which are automatically continuous (even norm decreasing). Such a $*$-homomorphism $\phi : A \to B$ is called *nondegenerate* if $\phi(A)B$ is dense in B. We denote the category of C*-algebras and nondegenerate $*$-homomorphisms between them by **C*Alg**. A good reference for general operator theory is [**Bla06**]. For C*-correspondences our main reference is [**RW98**], from which almost all our results can be derived. Also see the papers [**Lan01a**; **Lan01b**], or the more recent [**BMZ13**].

2.1 Hilbert C*-modules

The first appearance of Hilbert C*-modules was in the work of Kaplansky [**Kap53**], where he considered modules over function algebras of compact spaces. Twenty years later the general case for C*-algebras appeared in the work of Paschke [**Pas73**] and Rieffel [**Rie74a**]. Hilbert C*-modules form a generalisation of Hilbert spaces and complex-valued inner products. It turns out that the structure of a C*-algebra allows us to re-state the definition of an inner product almost verbatim. This is due in part to the behaviour of positive elements in a C*-algebra. Recall that an element $a \in A$ of a C*-algebra is called *positive* if it is of the form $a = b^*b$ for some $b \in A$. If A is unital, then a is positive if and only if its spectrum $\sigma(a)$ is contained in the non-negative reals.

If B is a complex associative algebra, recall that a vector space E is called a *right B-module* if it is equipped with a bilinear right action $\mathsf{E} \times B \to \mathsf{E}$. To emphasise this, we denote such a module by E_B.

Definition 2.1. Let B be a C*-algebra, and let E_B be a right B-module. A sesquilinear map (with the convention that it is linear in the second component) $\langle \cdot, \cdot \rangle_B : \mathsf{E} \times \mathsf{E} \to B$ is called a *B-valued inner product* on E if it satisfies the following four conditions:

1. The second component is B-linear in the sense that for all $\xi, \eta \in \mathsf{E}$ and $b \in B$ we have $\langle \xi, \eta \cdot b \rangle_B = \langle \xi, \eta \rangle_B b$, where on the right hand side the multiplication is that of B;
2. We have $\langle \xi, \eta \rangle_B^* = \langle \eta, \xi \rangle_B$ for any $\xi, \eta \in \mathsf{E}$;
3. For all $\xi \in \mathsf{E}$, the element $\langle \xi, \xi \rangle_B$ is positive in B;
4. There is a nondegeneracy condition: $\langle \xi, \xi \rangle_B = 0$ if and only if $\xi = 0$.

We say that the right B-module E_B equipped with a B-valued inner product $\langle \cdot, \cdot \rangle_B$ is a *right inner product B-module*. The inner product defines a norm $\|\cdot\|_{\mathsf{E}}$ on E by the formula $\|\xi\|_{\mathsf{E}} := \|\langle \xi, \xi \rangle_B\|_B^{1/2}$. If E is complete with respect to this norm, we say it is a *right Hilbert C*-module* (over B).

The notion of a *left Hilbert C*-module* (over A) is defined completely analogously, in which case we denote the inner product by a prescript: ${}_A\langle \cdot, \cdot \rangle$.

Given two Hilbert C*-modules E_B and F_B (both over B), whose B-valued inner products we denote by $\langle \cdot, \cdot \rangle_B^{\mathsf{E}}$ and $\langle \cdot, \cdot \rangle_B^{\mathsf{F}}$, a function $T : \mathsf{E} \to \mathsf{F}$ is called *adjointable* if there exists another function $T^* : \mathsf{F} \to \mathsf{E}$ such that for all $\xi \in \mathsf{E}$ and $\zeta \in \mathsf{F}$ we have the equality

$$\langle T(\xi), \zeta \rangle_B^{\mathsf{F}} = \langle \xi, T^*(\zeta) \rangle_B^{\mathsf{E}}.$$

From this defining relation it follows that any adjointable function is necessarily a bounded linear map, and has to respect the B-module structure in the sense that $T(\xi b) = T(\xi)b$ for all $\xi \in \mathsf{E}$ and $b \in B$. For this reason, we shall refer to adjointable functions as *adjointable operators*. We denote the space of adjointable operators from E to F by $\mathscr{L}_B(\mathsf{E}, \mathsf{F})$. This space forms the analogue of bounded operators on a Hilbert space:

Proposition 2.2 ([**RW98**, Proposition 2.21])**.** *Let E_B be a Hilbert C*-module. The space $\mathscr{L}_B(\mathsf{E})$ of adjointable operators on E is a C*-algebra with respect to the operator norm.*

Example 2.3. A Hilbert C*-module over $\mathbb{C}$ is just a Hilbert space $\mathscr{H}$. On a Hilbert space every operator is adjointable, so we have $\mathscr{L}_{\mathbb{C}}(\mathscr{H}) = \mathrm{B}(\mathscr{H})$, the C*-algebra of bounded operators on $\mathscr{H}$.

Besides adjointable operators on Hilbert C*-modules, we have the important class of *compact operators*. For two Hilbert C*-modules E_B and F_B we define the *finite rank* operators as[2] (spans of)

$$\theta_{\zeta,\xi} : \mathsf{E} \longrightarrow \mathsf{F}; \qquad \eta \longmapsto \zeta \cdot \langle \xi, \eta \rangle_B^{\mathsf{E}},$$

where $\xi \in \mathsf{E}$ and $\zeta \in \mathsf{F}$. All finite rank operators are adjointable, with $\theta_{\zeta,\xi}^* = \theta_{\xi,\zeta}$. The *compact operators* $\mathsf{E} \to \mathsf{F}$ are defined as the closed span

$$\mathrm{K}_B(\mathsf{E}, \mathsf{F}) := \overline{\mathrm{span}}\{\theta_{\zeta,\xi} : \zeta \in \mathsf{F}, \xi \in \mathsf{E}\}.$$

In analogy to the theory of Hilbert spaces:

Proposition 2.4 ([**RW98**, Lemma 2.25])**.** *Let E_B be a Hilbert C*-module. The space $\mathrm{K}_B(\mathsf{E})$ of compact operators on E is a closed two-sided ideal in the adjointable operators $\mathscr{L}_B(\mathsf{E})$, and hence a C*-algebra.*

An important construction that will actually be a crucial part of our later proofs is the following.

Example 2.5 ([**RW98**, Lemma 2.30])**.** Let E_A be a Hilbert C*-module. Then E has the structure of a left Hilbert C*-module over $\mathrm{K}_A(\mathsf{E})$ with the obvious action and inner product ${}_{\mathrm{K}_A(\mathsf{E})}\langle \xi, \zeta \rangle := \theta_{\xi,\zeta}$.

2.2 The bicategory of C*-algebras and C*-correspondences

Let us expand on some remarks made in the abstract, in order to motivate why (beyond, e.g., the representation theory of locally compact Hausdorff groups) the category **C*Alg** is unsatisfactory. In noncommutative geometry, geometric spaces are modelled using C*-algebras. Already in the case of finite-dimensional (matrix) algebras and finite discrete spaces we observe the following (see [**vSu15**, Chapter 2]). Every finite space X gives rise to a commutative matrix algebra $C(X)$, and every finite-dimensional matrix algebra A gives rise to a finite space $\widehat{A}$. It follows as a special case of Gelfand duality that when A is commutative we have $A \cong C(\widehat{A})$. But, of course, A may well not be commutative in general. In that case the assignment $A \mapsto \widehat{A}$ does not *reflect isomorphism*. That means there are *non*-isomorphic matrix algebras A and B, such that nevertheless $\widehat{A} \cong \widehat{B}$. This defect in the category **C*Alg** needs to be remedied if, as the theory of noncommutative geometry (very usefully) proposes, we want to do geometry with C*-algebras instead of spaces. One way of interpreting this defect is by saying **C*Alg** lacks the morphisms to induce the appropriate isomorphisms. The goal of the subsequent sections is, then, to describe a generalised notion of ∗-homomorphism, thereby adding morphisms (and hence isomorphisms) to **C*Alg**, which describe more accurately the topological and geometric aspects[3] of C*-algebras (cf. [**vSu15**, Theorem 2.14]).

[2] In Dirac notation, one would write $\theta_{\zeta,\xi} = |\zeta\rangle\langle\xi|$.

[3] There is an analogous situation in the theory of Lie groupoids and differentiable stacks, where the motivation to introduce Morita equivalence is even more apparent. This is beyond the scope of this note, but we refer to [**Blo08**].

We will not go into more detail on these questions of 'geometric properties' (which are also more subtle than we portray them here; see Note 8 in [vSu15, Page 23]), but will rather focus on the technical aspects of Morita equivalence itself. Namely, the generalised morphisms between C*-algebras (Definition 2.6) defines a bicategorical equivalence, while on the other hand we have Rieffel's more intrinsic definition of Morita equivalence (Definition 2.16). To prove that these notions coincide will be the content of the remainder of this note.

Definition 2.6. Let A and B be two C*-algebras. A *C*-correspondence* from A to B is a right Hilbert C*-module E_B together with a nondegenerate $*$-homomorphism $\phi : A \to \mathscr{L}_B(\mathsf{E})$. Sometimes we denote such correspondences by $(\mathsf{E}, \phi) : A \to B$, but more often implicitly by ${}_A\mathsf{E}_B$. (Nondegeneracy of $\phi : A \to \mathscr{L}_B(\mathsf{E})$ here means that the subspace $\phi(A)(\mathsf{E})$ is dense in E. Some authors omit this requirement.)

Example 2.7. As noted in Example 2.3, a Hilbert C*-module over $\mathbb{C}$ is just a Hilbert space $\mathscr{H}$. On a Hilbert space the adjointable operators are exactly the bounded operators, so $\mathscr{L}_{\mathbb{C}}(\mathscr{H}) = \mathrm{B}(\mathscr{H})$. Hence, a C*-correspondence from A to $\mathbb{C}$ is just a nondegenerate $*$-representation $A \to \mathrm{B}(\mathscr{H})$. We will see below that C*-algebras are also special cases of Hilbert C*-modules, and will play an important rôle in this paper as the identity elements of **C*Corr**.

The notation ${}_A\mathsf{E}_B$ suggests the structure of a bimodule. Indeed, we have a left action of A on E given by $a \cdot \xi := \phi(a)(\xi)$, and this action commutes with the one of B precisely because each $\phi(a)$ is a B-module map. That ϕ respects the involution moreover gives the identity $\langle a \cdot \xi, \eta\rangle_B = \langle \xi, a^* \cdot \eta\rangle_B$.

Lemma 2.8 ([Lan01b, Lemma 3.8])**.** *Let $(\mathsf{E}, \phi) : A \to B$ be a C*-correspondence. Nondegeneracy of ϕ implies (and is in fact equivalent to the property) that for every approximate unit $(e_i)_{i\in I}$ in A and $\xi \in \mathsf{E}$ we have $\lim_{i\in I} e_i\xi = \lim_{i\in I} \phi(e_i)(\xi) = \xi$.*

An important step in the construction of **C*Corr** is the definition of composition of C*-correspondences. This is done via the *balanced tensor product*, originally introduced by Rieffel in [Rie74a]. The construction is as follows (see [RW98, Proposition 3.16]): consider two C*-correspondences $(\mathsf{E}, \phi) : A \to B$ and $(\mathsf{F}, \psi) : B \to C$. The algebraic tensor product $\mathsf{E} \otimes^{\mathrm{alg}}_{\mathbb{C}} \mathsf{F}$ inherits a C-module structure by action on the right component: $(\xi \otimes \zeta)c := \xi \otimes (\zeta c)$, and the following turns it into an inner product C-module:

$$\langle\!\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2 \rangle\!\rangle_C := \langle \zeta_1, \langle \xi_1, \xi_2\rangle_B \zeta_2\rangle_C .$$

The vector space $\mathsf{E} \otimes_B \mathsf{F}$ is defined as the completion of the quotient of $\mathsf{E} \otimes^{\mathrm{alg}}_{\mathbb{C}} \mathsf{F}$ by $\mathrm{span}\{\xi \otimes \zeta : \langle\!\langle \xi \otimes \zeta, \xi \otimes \zeta\rangle\!\rangle_C = 0\}$ with respect to the norm induced by the C-valued inner product $\langle\!\langle \cdot, \cdot\rangle\!\rangle_C$. As such, we have identities $\xi b \otimes \zeta = \xi \otimes \psi(b)(\zeta) =: \xi \otimes b\zeta$. Note that the C-module structure carries over to $\mathsf{E} \otimes_B \mathsf{F}$, hence making it a right Hilbert C*-module over C.

Now, given an adjointable operator $T \in \mathscr{L}_B(\mathsf{E})$, the map

$$T \otimes 1 : \mathsf{E} \otimes_B \mathsf{F} \longrightarrow \mathsf{E} \otimes_B \mathsf{F} \qquad \text{defined on pure tensors by} \qquad \xi \otimes \zeta \longmapsto (T\xi) \otimes \zeta$$

is an adjointable operator: $T \otimes 1 \in \mathscr{L}_C(\mathsf{E} \otimes_B \mathsf{F})$. Moreover, the assignment $- \otimes 1 : \mathscr{L}_B(\mathsf{E}) \to \mathscr{L}_C(\mathsf{E} \otimes_B \mathsf{F})$ sending $T \mapsto T \otimes 1$ is a $*$-homomorphism.

Definition 2.9. The composition of the C*-correspondences (E, ϕ) and (F, ψ) is defined by taking the *balanced tensor product* $\mathsf{E} \otimes_B \mathsf{F}$ as a right Hilbert C*-module over C together with the nondegenerate $*$-homomorphism $\phi \otimes 1 : A \to \mathscr{L}_C(\mathsf{E} \otimes_B \mathsf{F})$ defined by $a \mapsto \phi(a) \otimes 1$. (One could write this composition cumbersomely as $(\mathsf{E} \otimes_B \mathsf{F}, (- \otimes 1) \circ \phi)) : A \to C$.)

As we alluded to in Example 2.7, any C*-algebra A can also be seen as a C*-correspondence on itself. As a vector space set $\mathsf{E} = A$, with right multiplication as the right A-module structure, and

$\langle \cdot, \cdot \rangle_A : A \times A \to A; (a, b) \mapsto a^* b$ as the inner product. The norm that this induces is equal to the norm of A; hence A_A is a Hilbert C*-module. Further, left multiplication $M_a : b \mapsto ab$ is clearly an adjointable map, and the inclusion $A \hookrightarrow \mathscr{L}_A(A); a \mapsto M_a$ is nondegenerate by the existence of approximate units.

Definition 2.10. The resulting C*-correspondence ${}_A A_A$ is called the *unit correspondence* of A.

The next definition is crucial and necessary in the definition of $\mathbf{C^*Corr}$. Namely, it is clear that the composition of C*-correspondences cannot be strictly associative, if only for the simple fact that the Cartesian product of sets $(X \times Y) \times Z$ is in general not equal to $X \times (Y \times Z)$. They are nevertheless bijective as sets, which hints at the fact that there may be some notion of equivalence that makes the composition of C*-correspondences associative in some weaker sense. This notion is the following:

Definition 2.11. Let (E, ϕ) and (F, ψ) be two C*-correspondences from A to B. A *unitary intertwiner* $u : (\mathsf{E}, \phi) \Rightarrow (\mathsf{F}, \psi)$ is a unitary adjointable map $u \in \mathscr{L}_B(\mathsf{E}, \mathsf{F})$ (i.e., $u^* = u^{-1}$) such that for all $a \in A$ the following diagram commutes (which just amounts to u being A-linear):

$$\begin{array}{ccc} \mathsf{E} & \xrightarrow{\phi(a)} & \mathsf{E} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathsf{F} & \xrightarrow[\psi(a)]{} & \mathsf{F}. \end{array}$$

The unitary intertwiners will be exactly the 2-morphisms in $\mathbf{C^*Corr}$. Note that, almost by definition, a unitary intertwiner between C*-correspondences is invertible (i.e., if u is a unitary intertwiner, then u^{-1}, going the other way around, is one as well). Therefore, whenever there exists a unitary intertwiner between C*-correspondences, we may consider them *unitarily isomorphic*. In other words, every 2-morphism in $\mathbf{C^*Corr}$ is a 2-*isomorphism*. With this, we can start to investigate the behaviour of composition of C*-correspondences up to unitary isomorphism. Whereas it was impossible for this composition to be strictly associative, or to allow strict identity morphisms, we will prove these properties do hold up to unitary isomorphism. Firstly, we can prove that the unit correspondences of Definition **2.10** behave like identity arrows:

Proposition 2.12. *Let A and B be C*-algebras, and consider a C*-correspondence ${}_A\mathsf{E}_B$. Then there are unitary isomorphisms*

$${}_A A \otimes_A \mathsf{E}_B \Longrightarrow {}_A\mathsf{E}_B, \qquad \textit{and} \qquad {}_A\mathsf{E} \otimes_B B_B \Longrightarrow {}_A\mathsf{E}_B.$$

Proof. Suppose that the left A-module structure of E arises from the nondegenerate $*$-homomorphism $\phi : A \to \mathscr{L}_B(\mathsf{E})$. Then define $u : A \otimes_A \mathsf{E} \to \mathsf{E}$ by the extension of left multiplication: $a \otimes \xi \mapsto \phi(a)(\xi) = a\xi$. This map is well defined because ϕ is a $*$-homomorphism:

$$u(ab \otimes \xi) = \phi(ab)(\xi) = \phi(a)(\phi(b)\xi) = u(a \otimes b\xi).$$

For its adjoint, we take an approximate unit $(e_i)_{i \in I}$ of A, and set $u^* : \mathsf{E} \to A \otimes_A \mathsf{E}$ as $\xi \mapsto \lim_{i \in I} e_i \otimes \xi$. To show that the limit on the right hand side exists, take $i, j \in I$ and calculate the norm:

$$\begin{aligned} \|e_i \otimes \xi - e_j \otimes \xi\|^2_{A \otimes_A \mathsf{E}} &= \|\langle\!\langle (e_i - e_j) \otimes \xi, (e_i - e_j) \otimes \xi \rangle\!\rangle_B\|_B \\ &= \big\|\langle \xi, \langle e_i - e_j, e_i - e_j \rangle_A \xi \rangle_B\big\|_B \\ &= \|(e_i - e_j)\xi\|^2_{\mathsf{E}} \leqslant \|\xi\|^2_{\mathsf{E}} \|\phi\|^2 \|e_i - e_j\|^2_A . \end{aligned}$$

It is clear from this approximation that the sequence $(e_i \otimes \xi)_{i\in I}$ is Cauchy in $A \otimes_A \mathsf{E}$, proving that the limit $\lim_{i\in I} e_i \otimes \xi$ exists. A straightforward calculation shows that u^* is the adjoint of u:

$$\begin{aligned}\langle u(a\otimes \xi_1), \xi_2\rangle_B &= \langle a\xi_1, \xi_2\rangle_B = \langle \xi_1, a^*\xi_2\rangle_B \\ &= \lim_{i\in I}\langle \xi_1, \langle a, e_i\rangle \xi_2\rangle_B = \lim_{i\in I}\langle\!\langle a\otimes\xi_1, e_i\otimes\xi_2\rangle\!\rangle_B \\ &= \langle\!\langle a\otimes \xi_1, \lim_{i\in I} e_i\otimes \xi_2\rangle\!\rangle_B = \langle\!\langle a\otimes\xi_1, u^*(\xi_2)\rangle\!\rangle_B.\end{aligned}$$

Lemma 2.8 ensures that u^* is a right-sided inverse of u:

$$u\circ u^*(\xi) = u\left(\lim_{i\in I} e_i\otimes\xi\right) = \lim_{i\in I} u(e_i\otimes\xi) = \lim_{i\in I} e_i\xi = \xi,$$

and the defining relation of the balanced tensor product gives

$$u^*\circ u(a\otimes\xi) = u^*(a\xi) = \lim_{i\in I} e_i\otimes(a\xi) = \lim_{i\in I}(e_i a)\otimes\xi = a\otimes\xi,$$

proving that u defines a unitary intertwiner ${}_A A\otimes_A \mathsf{E}_B \Rightarrow {}_A\mathsf{E}_B$, as desired.

The construction of ${}_A\mathsf{E}\otimes_B B_B \Rightarrow {}_A\mathsf{E}_B$ is analogous. Note that here we already enjoy a nondegeneracy property inherent to all Hilbert C*-modules, namely that $\mathsf{E}\cdot B$ is always dense in E by the Cauchy-Schwarz inequality. □

In a similar vein we have an associativity condition, whose proof is similar.

Proposition 2.13. *Consider three C*-correspondences ${}_A\mathsf{E}_B$, ${}_B\mathsf{F}_C$ and ${}_C\mathsf{G}_D$. Then there is a unitary isomorphism*

$$(\mathsf{E}\otimes_B \mathsf{F})\otimes_C \mathsf{G} \Longrightarrow \mathsf{E}\otimes_B(\mathsf{F}\otimes_C\mathsf{G})$$

defined on pure tensors by the obvious map $(\xi\otimes\zeta)\otimes\gamma\mapsto\xi\otimes(\zeta\otimes\gamma)$.

Summarising what we have so far (see also e.g. [Lan01a; EKQ06; Blo08; BMZ13]):

Theorem 2.14. *There is a bicategory* **C*Corr** *consisting of C*-algebras as objects, C*-correspondences as morphisms, and unitary intertwiners as 2-morphisms. The composition of C*-correspondences is via the balanced tensor product, and the unit correspondences are the identity arrows.*

To show that the framework of C*-correspondences subsumes the traditional notion of morphism between C*-algebras, we show that every nondegenerate $*$-homomorphism gives rise to a C*-correspondence:

Proposition 2.15 (Correspondisation)**.** *We have an inclusion functor ι :* **C*Alg** $\hookrightarrow$ **C*Corr** *(on the level of 1-categories) acting as identity on objects, and sending a nondegenerate $*$-homomorphism $\phi: A\to B$ to the C*-correspondence ${}_A\iota(\phi)_B$, where $\iota(\phi) = B$ carries the natural B-module structure, and A acts adjointably on B by $a\cdot b := \phi(a)b$.*

Proof. It is clear that $\iota(\phi)_B$ is a Hilbert C*-module with respect to the inner product $\langle b_1, b_2\rangle_B = b_1^* b_2$, as in the unit correspondence. The inclusion of $B\hookrightarrow \mathscr{L}_B(B)$, sending $b\in B$ to its left multiplication operator M_b, is a well-defined $*$-homomorphism between C*-algebras. The $*$-homomorphism $A\to\mathscr{L}_B(B)$ is then the composition of ϕ and this inclusion, i.e., $a\mapsto M_{\phi(a)}$. This is evidently nondegenerate whenever ϕ is. Hence ${}_A\iota(\phi)_B$ is a well-defined C*-correspondence from A to B.

Now suppose that we have two $*$-homomorphisms $\phi: A\to B$ and $\psi: B\to C$. We construct a unitary intertwiner

$$u: {}_A\iota(\phi)\otimes_B \iota(\psi)_C \Longrightarrow {}_A\iota(\psi\circ\phi)_C.$$

For this, set $u:\iota(\phi)\otimes_B\iota(\psi) = B\otimes_B C\to\iota(\psi\circ\phi) = C$ on pure tensors by $b\otimes c\mapsto\psi(b)c$. This is well defined on the balanced tensor product since ψ is a $*$-homomorphism. It completes to a unitary intertwiner by similar arguments as in the proof of Proposition 2.12 using the nondegeneracy of ψ. □

2.3 C*-correspondences and equivalence bimodules

Now that we have set the stage for C*-correspondences, we set out to prove our main theorem characterising the equivalences in **C*Corr** (Theorem **2.27**). Let us spell out what it means for C*-algebras A and B to be equivalent in **C*Corr**. Recall that by Definition **1.1** two objects in a bicategory are said to be equivalent when there is a weakly invertible arrow between them. In this case, A and B are equivalent in **C*Corr** if and only if there exist C*-correspondences ${}_A\mathsf{E}_B$ and ${}_B\mathsf{F}_A$ with unitary isomorphisms

$$ {}_A\mathsf{E} \otimes_B \mathsf{F}_A \Longrightarrow {}_AA_A \qquad \text{and} \qquad {}_B\mathsf{F} \otimes_A \mathsf{E}_B \Longrightarrow {}_BB_B. $$

We would like necessary and sufficient conditions inherent to a C*-correspondence ${}_A\mathsf{E}_B$ to determine whether it admits a weak inverse ${}_B\mathsf{F}_A$ or not. We will prove in Theorem **2.27** that the following definition will provide exactly that.

Definition 2.16 ([**Rie74b**, Definition 7.5])**.** An *equivalence bimodule* (also called an *imprimitivity bimodule* in [**RW98**, Definition 3.1]) between A and B is a bimodule ${}_A\mathsf{E}_B$ such that:

1. ${}_A\mathsf{E}$ is a *full* left Hilbert C*-module with respect to an A-valued inner product ${}_A\langle\cdot,\cdot\rangle$, and E_B is a full right Hilbert C*-module with respect to a B-valued inner product $\langle\cdot,\cdot\rangle_B$. Here *fullness* means ${}_A\langle\mathsf{E},\mathsf{E}\rangle$ is dense in A, and similarly for E_B;
2. The bimodule structure is adjointable in that for all $\xi, \eta \in \mathsf{E}$, $a \in A$ and $b \in B$ we have
$$ {}_A\langle\xi \cdot b, \eta\rangle = {}_A\langle\xi, \eta \cdot b^*\rangle \qquad \text{and} \qquad \langle a \cdot \xi, \eta\rangle_B = \langle\xi, a^* \cdot \eta\rangle_B; $$
3. Lastly, we have for all $\xi, \eta, \zeta \in \mathsf{E}$ a compatibility for the A- and B-valued inner products:
$$ {}_A\langle\xi, \eta\rangle \cdot \zeta = \xi \cdot \langle\eta, \zeta\rangle_B. $$

Note that the two norms on E, induced by ${}_A\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_B$, must agree as a consequence of the third condition.

Definition 2.17. Two C*-algebras are called *(strongly) Morita equivalent* if there exists an equivalence bimodule between them. (This was the original definition due to Rieffel [**Rie74b**].) It can be proved directly that this forms a genuine equivalence relation [**RW98**, Proposition 3.18].

Note that equivalence bimodules are, in particular, C*-correspondences. Indeed, that ${}_A\mathsf{E}$ is a left Hilbert C*-module makes it so that the $*$-homomorphism $\phi : A \to \mathscr{L}_B(\mathsf{E})$, sending $a \in A$ to the left module action $M_a : \xi \mapsto a\xi$, is nondegenerate.

Example 2.18. Every C*-algebra is Morita equivalent to itself through its unit correspondence.

We now have two notions of equivalence between C*-algebras. We have the Morita equivalence of Definition **2.17**, and the bicategorical equivalence in **C*Corr**. The goal of Theorem **2.27** is to prove that these two coincide. To start this proof, we show that the properties in Definition **2.16** provide sufficient conditions to define a weak inverse of a C*-correspondence. Given an equivalence bimodule ${}_A\mathsf{E}_B$, we construct a new bimodule ${}_B\overline{\mathsf{E}}_A$ as follows (cf. [**RW98**, p.49]). As a vector space, $\overline{\mathsf{E}}$ is the *conjugate* of E, meaning that the addition is the same, but the scalar multiplication is defined by $\lambda \cdot \xi := \overline{\lambda}\xi$, where on the left the multiplication is in $\overline{\mathsf{E}}$ and on the right hand side it is that of E. The bimodule structure of ${}_B\overline{\mathsf{E}}_A$ is then effectively flipping that of ${}_A\mathsf{E}_B$:

$$ \begin{aligned} b \cdot \xi &:= \xi b^*, \qquad & {}_B\langle\xi, \eta\rangle &:= \langle\xi, \eta\rangle_B; \\ \xi \cdot a &:= a^*\xi, \qquad & \langle\xi, \eta\rangle_A &:= {}_A\langle\xi, \eta\rangle, \end{aligned} $$

and it is obvious that it forms an equivalence bimodule. (Here, on the right hand sides, each time the bimodule operations of the original bimodule are understood.) It is obvious that if ${}_A\mathsf{E}_B$ is an equivalence bimodule, then so is ${}_B\overline{\mathsf{E}}_A$.

Proposition 2.19. *Let ${}_A\mathsf{E}_B$ be an equivalence bimodule. Then ${}_B\overline{\mathsf{E}} \otimes_A \mathsf{E}_B$ is an equivalence bimodule with inner products*

$$\langle\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2 \rangle\rangle_B := \langle \zeta_1 \langle \xi_1, \xi_2 \rangle_A, \zeta_2 \rangle_B \,,$$
$$_B\langle\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2 \rangle\rangle := \langle \xi_1 \langle \zeta_1, \zeta_2 \rangle_A, \xi_2 \rangle_B \,.$$

(A similar result holds for ${}_A\mathsf{E} \otimes_B \overline{\mathsf{E}}_A$.)

Proof. (Note that this is a special case of [**RW98**, Proposition 3.16], but the proof is significantly simpler here, and we won't need the general case.)

Fullness of ${}_B\overline{\mathsf{E}} \otimes_A \mathsf{E}_B$ follows from fullness of ${}_A\mathsf{E}_B$ and nondegeneracy of $a \mapsto M_a \in \mathscr{L}_B(\mathsf{E})$. Namely,

$$\langle\langle \overline{\mathsf{E}} \otimes_A \mathsf{E}, \overline{\mathsf{E}} \otimes_A \mathsf{E} \rangle\rangle_B = \big\langle \mathsf{E} \langle \overline{\mathsf{E}}, \overline{\mathsf{E}} \rangle_A, \mathsf{E} \big\rangle_B = \big\langle \mathsf{E}, {}_A\langle \mathsf{E}, \mathsf{E} \rangle \mathsf{E} \big\rangle_B$$

is dense in $\langle \mathsf{E}, A\mathsf{E} \rangle_B$ by fullness of ${}_A\mathsf{E}$. And since $A\mathsf{E}$ is dense in E, $\langle \mathsf{E}, A\mathsf{E} \rangle_B$ is in turn dense in $\langle \mathsf{E}, \mathsf{E} \rangle_B$, which is subsequently dense in B by fullness of E_B. Similarly we find that ${}_B\overline{\mathsf{E}} \otimes_A \mathsf{E}_B$ is full with respect to the left inner product. Simple calculations further show that these inner products satisfy the compatibility conditions of Definition **2.16**. □

Before we continue, we give useful sufficient conditions for a map between equivalence bimodules to be a unitary intertwiner.

Lemma 2.20. *Let ${}_A\mathsf{E}_B$ and ${}_A\mathsf{F}_B$ be two equivalence bimodules from A to B. A linear map $u : \mathsf{E} \to \mathsf{F}$ is a unitary intertwiner whenever it preserves the left A- and right B-inner products, and has dense range.*

Proof. This is an elementary argument, as in [**RW98**, Remark 3.27]. Since the norms on E and F are induced by the inner products, which u preserves, the map is isometric. Hence it must be injective and must have closed range. Therefore, if the range is dense, u must be surjective. This surjectivity, together with the equation

$$\langle u(\xi), u(\zeta b) \rangle_B = \langle \xi, \zeta b \rangle_B = \langle \xi, \zeta \rangle_B b = \langle u(\xi), u(\zeta) \rangle_B b = \langle u(\xi), u(\zeta) b \rangle_B,$$

gives that u is B-linear, so $u \in \mathscr{L}_B(\mathsf{E}, \mathsf{F})$. A similar argument with the left A-inner product gives that u is A-linear, and hence the diagram in Definition **2.11** commutes. □

Now we can prove that equivalence bimodules are weakly invertible in **C*Corr**, their weak inverse given by their conjugate.

Proposition 2.21. *Let ${}_A\mathsf{E}_B$ be an equivalence bimodule. Then we have unitary intertwiners*

$$ {}_A\mathsf{E} \otimes_B \overline{\mathsf{E}}_A \Longrightarrow {}_A A_A; \qquad \xi \otimes \eta \longmapsto {}_A\langle \xi, \eta \rangle,$$
$$ {}_B\overline{\mathsf{E}} \otimes_A \mathsf{E}_B \Longrightarrow {}_B B_B; \qquad \xi \otimes \eta \longmapsto \langle \xi, \eta \rangle_B.$$

In particular, ${}_B\overline{\mathsf{E}}_A$ is a weak inverse for ${}_A\mathsf{E}_B$.

Proof. This is proven in [**RW98**, Proposition 3.28]. We follow their argument, employing Lemma **2.20**.

We claim that the map $u : \overline{\mathsf{E}} \otimes_A \mathsf{E} \to B$ defined on pure tensors by $\xi \otimes \eta \mapsto \langle \xi, \eta \rangle_B$ forms a unitary intertwiner. It is straightforward to see from the properties of the inner product $\langle \cdot, \cdot \rangle_B$ that u is well-defined. We show, to satisfy the sufficient conditions of Lemma **2.20**, that u preserves the inner-products and has dense range.

Consider two pure tensors $\xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2 \in \overline{\mathsf{E}} \otimes_A \mathsf{E}$. We get:

$$\begin{aligned}\langle u(\xi_1 \otimes \zeta_1), u(\xi_2 \otimes \zeta_2)\rangle_B^B &= \langle \langle \xi_1, \zeta_1\rangle_B, \langle \xi_2, \zeta_2\rangle_B\rangle_B \\ &= \langle \zeta_1, \xi_1\rangle_B \langle \xi_2, \zeta_2\rangle_B \\ &= \langle \zeta_1, \xi_1\langle \xi_2, \zeta_2\rangle_B\rangle_B \\ &= \langle \zeta_1, \langle \xi_1, \xi_2\rangle_B \zeta_2\rangle_B ,\end{aligned}$$

where in the last step we use that left multiplication in $\overline{\mathsf{E}}$ is the same as right multiplication by the adjoint in E. But now on the right hand side we recognise the inner product of the balanced tensor product $\overline{\mathsf{E}} \otimes_A \mathsf{E}$ (Proposition **2.19**): first we can write $\langle \xi_1, \xi_2\rangle_B \zeta_2 = \zeta_2 \langle \xi_2, \xi_1\rangle_B$, which gives

$$\langle u(\xi_1 \otimes \zeta_1), u(\xi_2 \otimes \zeta_2)\rangle_B^B = \langle \zeta_1, \zeta_2\langle \xi_2, \xi_1\rangle_B\rangle_B = \langle \zeta_1\langle \xi_1, \xi_1\rangle_B, \zeta_2\rangle_B = \langle\!\langle \xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2\rangle\!\rangle_B,$$

showing that u preserves the right B-valued inner product. To show that u preserves the left A-valued inner we can do a similar calculation, which we leave to the reader. Note further that $u(\overline{\mathsf{E}} \otimes_A \mathsf{E}) = \langle \mathsf{E}, \mathsf{E}\rangle_B$, and since E_B is full, it follows that u has dense range. Thus by Lemma **2.20** it follows that u is a unitary intertwiner. That ${}_A\mathsf{E} \otimes_B \overline{\mathsf{E}}_A \Rightarrow {}_A A_A$ is a unitary intertwiner then follows after the observation that $\overline{\overline{\mathsf{E}}} = \mathsf{E}$. □

Therefore, any equivalence bimodule, seen as a C*-correspondence, is an equivalence in **C*Corr**. However, the construction of ${}_B\overline{\mathsf{E}}_A$ depends explicitly on the left A-valued inner product, and so cannot be performed for arbitrary C*-correspondences. The next two lemmas tell us for which C*-correspondences we *can* perform this inversion.

Lemma 2.22. *Let E_B be a full Hilbert C*-module. Then ${}_{\mathrm{K}_B(\mathsf{E})}\mathsf{E}_B$ is an equivalence bimodule between the compact operators on E and B, with the left inner product sending a pair of vectors to their corresponding finite rank operator: ${}_{\mathrm{K}_B(\mathsf{E})}\langle \xi, \zeta\rangle := \theta_{\xi,\zeta}$.*

Conversely, if ${}_A\mathsf{E}_B$ is an equivalence bimodule, then there exists a $*$-isomorphism $\phi : A \to \mathrm{K}_B(\mathsf{E})$ such that $\phi({}_A\langle \xi, \zeta\rangle) = {}_{\mathrm{K}_B(\mathsf{E})}\langle \xi, \zeta\rangle$.

Proof. We follow the proof of [**RW98**, Proposition 3.8]. Let E_B be a full Hilbert C*-module over B. By Example **2.5** we get a left Hilbert C*-module structure over the compact operators $\mathrm{K}_B(\mathsf{E})$. Moreover, it is immediate from definition of compact operators that it is full. That the second and third condition of Definition **2.16** are satisfied by ${}_{\mathrm{K}_B(\mathsf{E})}\mathsf{E}_B$ follows directly from the definitions.

Now suppose that ${}_A\mathsf{E}_B$ is an equivalence bimodule. Denote by $\phi : A \to \mathscr{L}_B(\mathsf{E})$ the nondegenerate $*$-homomorphism mapping $a \mapsto M_a$. Then from the compatibility of the inner products it follows that

$$\phi\left({}_A\langle \xi, \zeta\rangle\right)(\eta) = {}_A\langle \xi, \zeta\rangle \eta = \xi\langle \zeta, \eta\rangle_B = \theta_{\xi,\zeta}(\eta),$$

where $\theta_{\xi,\zeta}$ is the finite-rank compact operator as defined in Section **2.1**. Hence we find $\phi\left({}_A\langle \xi, \zeta\rangle\right) = {}_{\mathrm{K}_B(\mathsf{E})}\langle \xi, \zeta\rangle$. By [**Bla06**, Corollary II.5.1.2] we know the image of ϕ is closed in $\mathscr{L}_B(\mathsf{E})$. Together with the previous equation and the fact that ${}_A\mathsf{E}$ is full, we get that $\mathrm{im}(\phi) = \mathrm{K}_B(\mathsf{E})$. That leaves us to show ϕ is injective. But that $a = 0$ whenever $M_a = 0$ follows immediately from the existence of approximate units. □

Lemma 2.23 ([**Lan01b**, Definition 3.6])**.** *Let ${}_A\mathsf{E}_B$ be a C*-correspondence, defined by a nondegenerate $*$-homomorphism $\phi : A \to \mathscr{L}_B(\mathsf{E})$. Then ${}_A\mathsf{E}_B$ is an equivalence bimodule if and only if the following two conditions are satisfied.*

1. *The right Hilbert C*-module E_B is full, i.e., $\langle \mathsf{E}, \mathsf{E}\rangle_B$ is dense in B;*
2. *The nondegenerate $*$-homomorphism $\phi : A \to \mathscr{L}_B(\mathsf{E})$ induces an isomorphism $A \cong \mathrm{K}_B(\mathsf{E})$.*

Here the A-valued inner product is given by the finite rank operators: ${}_A\langle\xi,\eta\rangle := \phi^{-1}(\theta_{\xi,\eta})$.

Proof. This follows directly from Lemma 2.22. □

As a last step to characterise equivalences in $\mathbf{C^*Corr}$, we need to prove that all of them are equivalence bimodules, the converse implication being Proposition 2.21. For that we follow the proof of [EKQ06, Lemma 2.4]. Before we begin we need some more lemmas:

Lemma 2.24. *Let ${}_A\mathsf{E}_B$ be a C*-correspondence with nondegenerate $*$-homomorphism $\phi : A \to \mathscr{L}_B(\mathsf{E})$. We have $\phi(a) = \phi(b)$ if and only if for all $\xi,\eta \in \mathsf{E}$ we have $\langle a\xi,\eta\rangle_B = \langle b\xi,\eta\rangle_B$.*

Proof. Suppose $a, b \in A$ satisfy the condition that for all $\xi,\eta \in \mathsf{E}$ we have $\langle a\xi,\eta\rangle_B = \langle b\xi,\eta\rangle_B$ (the other implication is trivial). By linearity we have $\langle (a-b)\xi,\eta\rangle_B = 0$. Setting $\eta = (a-b)\xi$ we get that $\|(a-b)\xi\|_{\mathsf{E}} = 0$ for all $\xi \in \mathsf{E}$, from which it follows that $\phi(a-b) = 0$. □

Lemma 2.25. *Let E_B be a Hilbert C*-module. For each $\xi \in \mathsf{E}$ we denote $L_\xi : B \to \mathsf{E}; b \mapsto \xi b$ and $D_\xi : \mathsf{E} \to B; \eta \mapsto \langle\xi,\eta\rangle_B$. Then $L : \mathsf{E} \to \mathrm{K}_B(B,\mathsf{E})$ mapping $\xi \mapsto L_\xi$ is an isometric linear isomorphism, and for each $\xi \in \mathsf{E}$ we have $L^*_\xi = D_\xi$.*

The proof is elementary, and we refer to [RW98, Lemma 2.32]. In particular, for $\xi,\eta \in \mathsf{E}$ we have $L_\xi \circ D_\eta = \theta_{\xi,\eta}$. It follows that $\mathrm{K}_B(B,\mathsf{E})\,\mathrm{K}_B(B,\mathsf{E})^*$, by which we mean the linear span of all elements of the form $T \circ S^*$ where $T, S \in \mathrm{K}_B(B,\mathsf{E})$, is dense in $\mathrm{K}_B(\mathsf{E})$. In light of the previous Lemma 2.25 we will write this symbolically as $\overline{\mathsf{E}\mathsf{E}^*} = \mathrm{K}_B(E)$.

Proposition 2.26. *Every equivalence in $\mathbf{C^*Corr}$ is an equivalence bimodule.*

Proof. (Again, note that we are following [EKQ06].) Let ${}_A\mathsf{E}_B$ be an equivalence, i.e., a C*-correspondence admitting a weak inverse ${}_B\mathsf{F}_A$, realised, say, by unitary isomorphisms

$$u_A : {}_A\mathsf{E} \otimes_B \mathsf{F}_A \Longrightarrow {}_A A_A \qquad \text{and} \qquad u_B : {}_B\mathsf{F} \otimes_A \mathsf{E}_B \Longrightarrow {}_B B_B.$$

We show that ${}_A\mathsf{E}_B$ satisfies both conditions of Lemma 2.23. It is immediate from the definition of the B-valued inner product on $\mathsf{F} \otimes_A \mathsf{E}$ that its image is contained in the B-valued inner product of E: $\langle\!\langle \mathsf{F}\otimes_A \mathsf{E}, \mathsf{F}\otimes_A \mathsf{E}\rangle\!\rangle_B \subseteq \langle \mathsf{E},\mathsf{E}\rangle_B$. But since u_B is unitary, we have

$$\langle\!\langle \mathsf{F}\otimes_A \mathsf{E}, \mathsf{F}\otimes_A \mathsf{E}\rangle\!\rangle_B = \langle u_B(\mathsf{F}\otimes_A \mathsf{E}), u_B(\mathsf{F}\otimes_A \mathsf{E})\rangle_B = \langle B, B\rangle_B,$$

which is obviously dense in B by the existence of approximate units. It follows that E_B is full.

The more difficult part is proving that $\phi : A \to \mathscr{L}_B(\mathsf{E})$ is an isomorphism from A into the compact operators on E. First note that a straightforward approximate unit argument shows that the inclusion $A \hookrightarrow \mathscr{L}_A(A); a \mapsto M_a$ is injective. The intertwiner u_A induces a $*$-isomorphism $\mathscr{L}_A(A) \cong \mathscr{L}_A(\mathsf{E}\otimes_B \mathsf{F})$ sending M_a to $u_A^* \circ M_a \circ u_A = \phi(a) \otimes 1$. This proves that the nondegenerate $*$-homomorphism $\phi \otimes 1 : A \to \mathscr{L}_A(\mathsf{E}\otimes_B \mathsf{F})$ (cf. Definition 2.9) is injective, too. Now, if $a \in \ker(\phi)$ it follows directly that $\phi(a)\otimes 1 = 0$, since $(\phi(a)\otimes 1)(\xi\otimes\zeta) = \phi(a)(\xi)\otimes\zeta = 0\otimes\zeta = 0$ for all pure tensors $\xi\otimes\zeta \in \mathsf{E}\otimes_B \mathsf{F}$. So the injectivity of $\phi\otimes 1$ carries over to ϕ.

This leaves us to prove that ϕ maps surjectively onto $\mathrm{K}_B(\mathsf{E})$. Define

$$u : \mathsf{F} \longrightarrow \mathscr{L}_B(\mathsf{E}, B); \qquad u(\zeta)(\xi) := u_B(\zeta\otimes\xi),$$

where $u_B : \mathsf{F}\otimes_A \mathsf{E} \to B$ is our unitary intertwiner. This makes u a well defined linear map. We will describe both the image $\phi(A)$ and $\mathrm{K}_B(\mathsf{E})$ in terms of u. First, we claim $u(\zeta_1)^* \circ u(\zeta_2) = \phi(\langle\zeta_1,\zeta_2\rangle^{\mathsf{F}}_B)$ for all $\zeta_1, \zeta_2 \in \mathsf{F}$. To this end, calculate:

$$\begin{aligned}
\langle u(\zeta_1)^* \circ u(\zeta_2)(\xi), \eta) \rangle_B^{\mathsf{E}} &= \langle u(\zeta_2)(\xi), u(\zeta_1)(\eta) \rangle_B^B \\
&= \langle u_B(\zeta_2 \otimes \xi), u_B(\zeta_1 \otimes \eta) \rangle_B^B \\
&= \langle\!\langle \zeta_2 \otimes \xi, \zeta_1 \otimes \eta \rangle\!\rangle_B^{\mathsf{F} \otimes_A \mathsf{E}} \\
&=: \langle \xi, \langle \zeta_2, \zeta_1 \rangle_A^{\mathsf{F}} \eta \rangle_B^{\mathsf{E}} \\
&= \langle \langle \zeta_1, \zeta_2 \rangle_A^{\mathsf{F}} \xi, \eta \rangle_B^{\mathsf{E}} = \langle \phi(\langle \zeta_1, \zeta_2 \rangle_A^{\mathsf{F}})(\xi), \eta \rangle_B^{\mathsf{E}},
\end{aligned}$$

so the claim follows by Lemma **2.24**. In particular, this gives $u(\mathsf{F})^* u(\mathsf{F}) = \phi(\langle \mathsf{F}, \mathsf{F} \rangle_A^{\mathsf{F}})$, and since F_A is full (this follows by the exact same argument as above to show that E_B is full) and ϕ is continuous, we get

$$\overline{u(\mathsf{F})^* u(\mathsf{F})} = \overline{\phi(\langle \mathsf{F}, \mathsf{F} \rangle_A^{\mathsf{F}})} = \phi\left(\overline{\langle \mathsf{F}, \mathsf{F} \rangle_A^{\mathsf{F}}}\right) = \phi(A). \tag{1}$$

The fact that u_B is an intertwiner gives for each $b \in B$ a commutative diagram

$$\begin{array}{ccc}
\mathsf{F} \otimes_A \mathsf{E} & \xrightarrow{\psi(b) \otimes 1} & \mathsf{F} \otimes_A \mathsf{E} \\
{\scriptstyle u_B} \big\downarrow & & \big\downarrow {\scriptstyle u_B} \\
B & \xrightarrow[M_b]{} & B,
\end{array}$$

where $\psi : B \to \mathscr{L}_A(\mathsf{F})$ is the nondegenerate $*$-homomorphism belonging to ${}_B\mathsf{F}_A$. This commutative diagram allows us to establish

$$\begin{aligned}
u(b\zeta)(\xi) &= u_B(b\zeta \otimes \xi) \\
&= u_B \circ (\psi(b) \otimes 1)(\zeta \otimes \xi) \\
&= M_b \circ u_B(\zeta \otimes \xi) = bu(\zeta)(\xi),
\end{aligned}$$

where $b \in B$, $\xi \in \mathsf{E}$ and $\zeta \in \mathsf{F}$. In other words, u is B-linear. Using this equation, we further calculate that for any given $c \in B$:

$$\begin{aligned}
\langle u(\zeta)^* \circ M_b(c), \xi \rangle_B^{\mathsf{E}} &= \langle u(\zeta)^*(bc), \xi \rangle_B^{\mathsf{E}} \\
&= \langle bc, u(\zeta)(\xi) \rangle_B^B \\
&= \langle c, b^* u(\zeta)(\xi) \rangle_B^B \\
&= \langle cu(b^*\zeta)(\xi) \rangle_B^B \\
&= \langle u(b^*\zeta)^*(a), \xi \rangle_B^{\mathsf{E}}.
\end{aligned}$$

It follows by a second employment of Lemma **2.24** that $u(\zeta)^* \circ M_b = u(b^*\zeta)^*$ as adjointable operators $B \to \mathsf{E}$. Noticing that $B^* = B$ and that multiplication operators M_b lie densely in $\mathrm{K}_B(B)$, we get

$$\overline{u(\mathsf{F})^* \, \mathrm{K}_B(B)} = u\left(\overline{\psi(B)\mathsf{F}}\right)^* = u(\mathsf{F})^*$$

since ψ is nondegenerate. And since ϕ is nondegenerate we then get using (1) that

$$\mathsf{E} = \overline{\phi(A)\mathsf{E}} = \overline{u(\mathsf{F})^* u(\mathsf{F})\mathsf{E}} = \overline{u(\mathsf{F})^* u_B(\mathsf{F} \otimes_A \mathsf{E})} = \overline{u(\mathsf{F})^* \, \mathrm{K}_B(B)} = u(\mathsf{F})^*.$$

Finally, the proof is finished by using the identification $\overline{\mathsf{E}\mathsf{E}^*} = \mathrm{K}_B(E)$, the equation above and (1), to obtain

$$\mathrm{K}_B(\mathsf{E}) = \overline{\mathsf{E}\mathsf{E}^*} = \overline{u(\mathsf{F})^* u(\mathsf{F})} = \phi(A). \qquad \square$$

Combination of Proposition **2.21** and Proposition **2.26** yields our ultimate result:

Theorem 2.27. *A C^*-correspondence ${}_A\mathsf{E}_B$ is an equivalence bimodule if and only if it is an equivalence in* $\mathbf{C^*Corr}$*. In other words, there exists a C^*-correspondence ${}_B\mathsf{F}_A$ together with with unitary isomorphisms*

$$ {}_A\mathsf{E} \otimes_B \mathsf{F}_A \Longrightarrow {}_A A_A \qquad \text{and} \qquad {}_B\mathsf{F} \otimes_A \mathsf{E}_B \Longrightarrow {}_B B_B $$

if and only if ${}_A\mathsf{E}_B$ is an equivalence bimodule.

Thus we have the conclusion of this note:

Corollary 2.28. *Two C^*-algebras are Morita equivalent in the sense of Definition **2.17** if and only if they are equivalent as objects in the bicategory* $\mathbf{C^*Corr}$.

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