Twisted Group C*-algebras and Projective Unitary Representations

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1 Introduction

Ordinary (non-twisted) group C*-algebras can be used to study the unitary representations of locally compact Hausdorff groups. In particular, if G is such a group, then its unitary representations are classified by non-degenerate $*$-representations of $C^*(G)$.

In physics, however, we are rather interested in *projective* unitary representations. For instance, elementary particles can be thought of as the irreducible projective unitary representations of the pertinent physical symmetry group (e.g., the Poincaré group). This group is almost always a (connected) Lie group. In that case, technical results like Lie's Third Theorem (also called the Cartan-Lie Theorem: every finite dimensional Lie algebra arises from a connected, simply connected Lie group) allow us to classify the projective unitary representations in terms of certain unitary representations of the centrally extended universal covering group.

For arbitrary locally compact Hausdorff groups, this machinery no longer works, and we have to resort to other techniques. In this essay we review the theory of *twisted* group C*-algebras. These generalise the ordinary group C*-algebras, and will turn out to help classify the projective unitary representations of a second countable locally compact Hausdorff group. This classification will be the subject of Section 3.

In [Lan98, Sections III.1.7-8], the theory of twisted group C*-algebras for globally smooth multipliers of unimodular Lie groups is developed. An older reference is [EL69], where twisted Banach $*$-algebras are constructed (see Section 3.1). To some extent we follow these references. (See also [RA15, Chapter 23].)

In the final section we outline some results that are involved in a twisted version of the Peter-Weyl theorem: for compact topological groups the twisted group C*-algebra can be decomposed as a direct sum over of matrix

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algebras, taken over all (classes of) irreducible projective representations of the group. See [Wil07, Proposition 3.4] for the ordinary version of this theorem, and [Lan98, Theorem III.1.8.1] for the smooth version of the twisted result.

Throughout the entire text we make great use of the lecture notes [AC18].

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2 Projective unitary representations and group cohomology

Projective unitary representations arise from quantum theory[1]. The pure states of a physical system may be represented by the vectors in some Hilbert space $\mathscr{H}$. If $\psi \in \mathscr{H}$ represents some physical state, it is well-known that any non-zero scalar multiple $\lambda\psi$ represents the same state. It is therefore that the state space of the physical system is rather the *projective Hilbert space* $\mathrm{P}(\mathscr{H})$. Whereas unitary operators represent the symmetries of a Hilbert space, the so-called *projective automorphisms*, preserving the *transition probability*, represent those of the projective Hilbert space. It was proved by Wigner in [Wig31] that

Theorem 2.1 (Wigner)**.** *Every projective automorphism T on $\mathrm{P}(\mathscr{H})$ arises from either a unitary or an anti-unitary operator U on $\mathscr{H}$, and U is determined uniquely by T up to a complex phase. (Cf. [Sch08, Theorem 3.3].)*

In this way, the unitary and anti-unitary operators on the Hilbert space $\mathscr{H}$ completely determine the automorphisms of the projective Hilbert space $\mathrm{P}(\mathscr{H})$. The ones that arise form the unitary transformations form the so called *projective unitary group*, and it can be realised in the following way. Let $\mathrm{U}(1)$ denote the circle group, containing all complex numbers of unit modulus. This group lives inside of the unitary group $\mathrm{U}(\mathscr{H})$ via the following injection: $\mathrm{diag} : \mathrm{U}(1) \to \mathrm{U}(\mathscr{H}); z \mapsto z\,\mathrm{id}_{\mathscr{H}}$. The *projective unitary group* of $\mathscr{H}$ is then defined as the quotient $\mathrm{PU}(\mathscr{H}) := \mathrm{U}(\mathscr{H})/\,\mathrm{diag}(\mathrm{U}(1))$. In other words, the projective unitary group contains equivalence classes of unitary operators that differ by complex phase.

2.1 The definition of projective unitary representations

Given a Hilbert space $\mathscr{H}$, we endow its unitary group $\mathrm{U}(\mathscr{H})$ with the strong operator topology, making it onto a topological group. The projective unitary group $\mathrm{PU}(\mathscr{H}) \cong \mathrm{U}(\mathscr{H})/\,\mathrm{U}(1)$ inherits a topological group structure from $\mathrm{U}(\mathscr{H})$ (see [Sch08, Proposition 3.11] and surrounding text).

Definition 2.2. A ***unitary representation*** of a topological group G is a continuous group homomorphism $G \to \mathrm{U}(\mathscr{H})$, where $\mathscr{H}$ is some Hilbert space. A ***projective unitary representation*** of G is a continuous homomorphism $G \to \mathrm{PU}(\mathscr{H})$.

A projective unitary representation can be thought of as a function $\pi : G \to \mathrm{U}(\mathscr{H})$, together with another function $\omega : G \times G \to \mathrm{U}(1) \cong \mathrm{diag}(\mathrm{U}(1))$, such that for all $x, y \in G$ we have the following *twisted* multiplicativity law:

$$\pi(x)\pi(y) = \omega(x, y)\pi(xy).$$

The function ω, due to the associativity of the group law, has to satisfy the following *cocycle conditions*:

$$\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z) \quad \text{and} \quad \omega(1_G, 1_G) = 1.$$

Indeed, the function ω is a *2-cocycle* on G with values in $\mathrm{U}(1)$. The projective representation π determines the cohomology class $[\omega]$ in the second group cohomology of G with values in $\mathrm{U}(1)$ uniquely. We refer to [Men17] for more details on these aspects.

For us, the working definition of projective representations will be the following:

[1] Besides physical motivation, there is also some mathematical incentive to study projective representations: the *'Mackey obstruction'*. See [Ros94, p.157].

Definition 2.3. Let G be a topological group. A continuous map $\omega : G \times G \to \mathrm{U}(1)$ that satisfies the cocycle conditions is called a ***multiplier***.

For a fixed multiplier ω, an ***ω-twisted unitary representation*** (or *ω-representation* for short) of G on some separable Hilbert space $\mathscr{H}$ is a continuous map $\pi : G \to \mathrm{U}(\mathscr{H})$ that satisfies $\pi(x)\pi(y) = \omega(x, y)\pi(xy)$ for all $x, y \in G$.

From the definitions it immediately follows that, for instance, $\omega(1_G, x) = \omega(x, 1_G) = 1$, and $\pi(1_G) = \mathrm{id}_{\mathscr{H}}$.

We say that an ω_1-representation $\pi_1 : G \to \mathrm{U}(\mathscr{H}_1)$ and and ω_2-representation $\pi_2 : G \to \mathrm{U}(\mathscr{H}_2)$ are ***(linearly) equivalent*** (or *isomorphic*) if there exists a bounded linear map $t : \mathscr{H}_1 \to \mathscr{H}_2$ (often called an *intertwiner*) such that for all $x \in G$ we have $t \circ \pi_1(x) = \pi_2(x) \circ t$. It follows, in fact, that the two multipliers ω_1 and ω_2 must in that case be equal.

The ***twisted unitary dual*** $\hat{G}_\omega$, with respect to some fixed multiplier ω, is defined as the space of all equivalence classes of irreducible ω-representations of G (on separable Hilbert spaces). Here an ω-representation $\pi : G \to \mathrm{U}(\mathscr{H})$ is called *irreducible* when the induced action on the projective Hilbert space $\mathrm{P}(\mathscr{H})$ has no non-trivial invariant subspaces. (Note that by [Lan98, Proposition III.1.5.1] the irreducible ω-representations of G arise from the irreducible unitary representations of a certain central extension G_ω of G by $\mathrm{U}(1)$.)

3 The twisted group C*-algebra

In this section we construct the twisted version of the group C*-algebra, where $\omega : G \times G \to \mathrm{U}(1)$ is some fixed measurable multiplier of some second countable locally compact Hausdorff group G. Note that ω is automatically bounded, since it takes values on the circle.

3.1 Construction of the twisted group algebra

On the space of integrable functions $L^1(G)$ (with respect to our left Haar measure μ on G) we define the following operation:

$$(f *_\omega g)(x) := \int_G \omega(y, y^{-1}x) f(y) g(y^{-1}x) \,\mathrm{d}\mu(y),$$

called the ***twisted convolution***, for any $f, g \in L^1(G)$ and $x \in G$. Since the multiplier is measurable, it follows that $f *_\omega g \in L^1(G)$, by similar arguments used to the un-twisted case (see [AC18, Theorem 1, 12-10-2017] and [Wil07, Footnote 17, pp.22-23]). In particular we have $\|f *_\omega g\|_1 = \|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$.

Proposition 3.1. *The twisted convolution turns $L^1(G)$ into an associative algebra, denoted $L^1_\omega(G)$.*

Proof. Distributivity and compatibility with scalar multiplication are obvious from linearity of the integral. We therefore only prove associativity. Consider three functions $f, g, h \in L^1(G)$, and some point $x \in G$ in the group. We have that

$$\begin{aligned}[(f *_\omega g) *_\omega h](x) &= \int_G \omega(y, y^{-1}x)(f *_\omega g)(y) h(y^{-1}x) \,\mathrm{d}\mu(y) \\ &= \int_G \int_G \omega(y, y^{-1}x)\omega(z, z^{-1}y) f(z) g(z^{-1}y) h(y^{-1}x) \,\mathrm{d}\mu(z) \,\mathrm{d}\mu(y).\end{aligned}$$

Using Fubini's Theorem to switch the integration order, we find that the above expression is equal to

$$\int_G f(z) \left(\int_G \omega(y, y^{-1}x)\omega(z, z^{-1}y) g(z^{-1}y) h(y^{-1}x) \,\mathrm{d}\mu(y) \right) \mathrm{d}\mu(z),$$

which in turn gives the equality

$$[(f *_\omega g) *_\omega h](x) = \int_G \int_G \omega(zy, (zy)^{-1}x)\omega(z, y) f(z) g(y) h((zy)^{-1}x) \,\mathrm{d}\mu(y) \,\mathrm{d}\mu(z)$$

by using left invariance of the Haar measure on the second integral, with a translation of z^{-1}. On the other hand, we have

$$\begin{aligned}[f *_\omega (g *_\omega h)](x) &= \int_G \omega(z, z^{-1}x) f(z)(g *_\omega h)(z^{-1}x) \, \mathrm{d}\mu(z) \\ &= \int_G \int_G \omega(z, z^{-1}x)\omega(y, (zy)^{-1}x) f(z) g(y) h((zy)^{-1}x) \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(z).\end{aligned}$$

Comparing the expressions, we see that (up to integration order) the only apparent difference between the convolutions is the multiplier term. However, a straightforward calculation with the cocycle condition ensures that these terms are equal: $\omega(z, z^{-1}x)\omega(y, (zy)^{-1}x) = \omega(zy, (zy)^{-1}x)\omega(z, y)$, showing that $[f *_\omega (g *_\omega h)](x) = [(f *_\omega g) *_\omega h](x)$ for all $x \in G$, and hence that the twisted convolution is associative. □

Proposition 3.2. *On $L^1_\omega(G)$, the following map is an involution:*

$$\begin{gathered} L^1_\omega(G) \to L^1_\omega(G) : f \mapsto f^\omega, \\ f^\omega(x) := \overline{\omega(x, x^{-1})}\Delta(x^{-1})\overline{f(x^{-1})}. \end{gathered}$$

Proof. First note that, since $\Delta : G \to \mathbb{R}^\times_{>0}$ is continuous, it is measurable with respect to the Haar measure. Now f^ω is the product of measurable functions, so it is also measurable. Moreover,

$$\|f^\omega\|_1 = \int_G \left|\overline{\omega(x, x^{-1})}\right| \left|\Delta(x^{-1})\overline{f(x^{-1})}\right| \mathrm{d}\mu(x) = \int_G \left|\Delta(x^{-1})\overline{f(x^{-1})}\right| \mathrm{d}\mu(x) = \|f^*\|_1,$$

where f^* is the un-twisted involution. Hence $f^\omega \in L^1(G)$, showing that the map is well-defined.

We now verify that the map has the properties of an involution. Conjugate linearity is obvious from the defining formula. Consider some $f \in L^1_\omega(G)$. Then, for all $x \in G$, we have

$$\begin{aligned}(f^\omega)^\omega(x) &= \overline{\omega(x, x^{-1})}\Delta(x^{-1})\overline{f^\omega(x^{-1})} \\ &= \overline{\omega(x, x^{-1})}\Delta(x^{-1})\overline{\left(\overline{\omega(x^{-1}, x)}\Delta(x)\overline{f(x)}\right)} \\ &= \overline{\omega(x, x^{-1})}\omega(x^{-1}, x) f(x),\end{aligned}$$

where we have used that the modular function Δ is a real-valued homomorphism. The cocycle condition for the multiplier gives the equality $\omega(x, x^{-1}) = \omega(x, x^{-1})\omega(1_G, x) = \omega(x, 1_G)\omega(x^{-1}, x) = \omega(x^{-1}, x)$. Because the multiplier maps into the unit circle, we therefore have

$$(f^\omega)^\omega(x) = \left|\omega(x, x^{-1})\right|^2 f(x) = f(x),$$

showing that $(f^\omega)^\omega = f$.

Consider another function $g \in L^1_\omega(G)$. We calculate, by using the definition:

$$(f *_\omega g)^\omega(x) = \Delta(x^{-1}) \int_G \overline{\omega(x, x^{-1})\omega(y, y^{-1}x^{-1}) f(y) g(y^{-1}x^{-1})} \, \mathrm{d}\mu(y). \tag{1}$$

On the other hand, we have, also by definition:

$$\begin{aligned}(g^\omega *_\omega f^\omega)(x) &= \int_G \omega(y, y^{-1}x) g^\omega(y) f^\omega(y^{-1}x) \, \mathrm{d}\mu(y) \\ &= \int_G \omega(y, y^{-1}x)\overline{\omega(y, y^{-1})}\Delta(y^{-1})\overline{g(y^{-1})\omega(y^{-1}x, x^{-1}y)}\Delta(x^{-1}y)\overline{f(x^{-1}y)} \, \mathrm{d}\mu(y).\end{aligned}$$

Using the commutativity of complex numbers and the fact that Δ is a homomorphism, we find that the terms involving the modular function simplify to $\Delta(y^{-1})\Delta(x^{-1}y) = \Delta(x^{-1}y)\Delta(y^{-1}) = \Delta(x^{-1})$. Applying a left translation of x^{-1} gives

$$(g^\omega *_\omega f^\omega)(x) = \Delta(x^{-1}) \int_G \omega(xy, y^{-1})\overline{\omega(xy, y^{-1}x^{-1})\omega(y^{-1}, y)f(y)g(y^{-1}x^{-1})}\, \mathrm{d}\mu(y).$$

Comparing to the expression for $(f *_\omega g)^\omega$ in (1), we see that the only apparent difference lies in the multiplier expressions. The cocycle identity gives $\omega(x, x^{-1})\omega(y, y^{-1}x^{-1}) = \omega(xy, y^{-1}x^{-1})\omega(x, y)$. Substituting this into (1) shows that it suffices to prove the following equality: $\overline{\omega(y^{-1}, y)}\omega(xy, y^{-1}) = \overline{\omega(x, y)}$. The cocycle condition gives $\omega(xy, y^{-1})\omega(x, y) = \omega(x, 1_G)\omega(y, y^{-1}) = \omega(y, y^{-1})$, which rewrites to $\omega(xy, y^{-1}) = \omega(y, y^{-1})/\omega(x, y)$. Using the fact that for elements on the circle the complex conjugate and inverse coincide, we find

$$\overline{\omega(y^{-1}, y)}\omega(xy, y^{-1}) = \frac{1}{\omega(y^{-1}, y)} \frac{\omega(y, y^{-1})}{\omega(x, y)} = \overline{\omega(x, y)}, \tag{2}$$

as desired. The equality $(f *_\omega g)^\omega = (g^\omega *_\omega f^\omega)$ follows, and we conclude that we have defined an involution on $L^1_\omega(G)$. □

We further conclude:

Proposition 3.3. *$L^1_\omega(G)$ is a Banach $*$-algebra. (See [EL69, Theorem 1].)*

3.2 The twisted group C*-algebra for continuous multipliers

The goal of this section is to complete the twisted convolution algebra $L^1_\omega(G)$ into a C*-algebra. The resulting C*-algebra is called the *twisted group C*-algebra*, and is denoted $C^*_\omega(G)$. Our starting point for this construction will be the space of compactly supported complex-valued continuous functions on G, denoted $C_c(G)$. It is well-known that $C_c(G)$ is a dense linear subspace of $L^1(G)$ with respect to the L^1-norm. The construction goes hand-in-hand with the proof of our main result in this section: there is a bijective correspondence between ω-representations $G \to \mathrm{U}(\mathscr{H})$ and nondegenerate $*$-representations $C^*_\omega(G) \to \mathrm{B}(\mathscr{H})$.

Firstly, we have the following proposition, whose proof is a straightforward generalisation of the un-twisted statement:

Proposition 3.4. *The space of compactly supported continuous functions $C_c(G)$ is closed under the twisted convolution and twisted involution. That is, if $f, g \in C_c(G)$, then $f *_\omega g \in C_c(G)$ and $f^\omega \in C_c(G)$.*

We denote by $C_\omega(G) \subseteq L^1_\omega(G)$ the involution subalgebra obtained from $C_c(G)$ in this way.

In order to relate the representation theory of G to that of $C_\omega(G)$ (and later to that of the twisted group C*-algebra), we make use of the following result:

Lemma 3.5. *Given an ω-representation $\pi : G \to \mathrm{U}(\mathscr{H})$, a function $f \in C_\omega(G)$ and a vector $\xi \in \mathscr{H}$, there exists a unique vector $\pi_C(f)\xi \in \mathscr{H}$ such that for all $\eta \in \mathscr{H}$ we have*

$$\langle \eta | \pi_C(f)\xi \rangle = \int_G \langle \eta | f(x)\pi(x)\xi \rangle \, \mathrm{d}\mu(x).$$

Proof. The construction of $\pi_C(f)\xi$ is independent on the multiplicative nature of π (i.e., independent on ω). We therefore refer to [AC18, Lemma 1, 16-10-2017] for a proof where π is an ordinary unitary representation. □

This lemma allows us to define a function $\pi_C : C_\omega(G) \to \mathrm{B}(\mathscr{H})$, called the ***integrated form*** of π. We claim that these integrated forms are actually $*$-homomorphisms. Note that $\|\pi_C(f)\| \leqslant \|f\|_1$, as can be seen from the fact that $\pi(x)$ is unitary.

To simplify the proof of this claim, we first consider a special case of an ω-representation:

Definition 3.6. The ***twisted left regular representation*** $L^\omega : G \to \mathrm{U}(L^2(G)); x \mapsto L^\omega_x$ of G is defined by the twisted left translation operators:

$$L^\omega_z f(x) := \omega(z, z^{-1}x) f(z^{-1}x).$$

The ***twisted right regular representation*** $R^\omega : G \to \mathrm{U}(L^2(G)); x \mapsto R^\omega_x$ is defined by the right twisted translation operators:

$$R^\omega_z f(x) := \omega(x, z) f(xz).$$

Proposition 3.7. *The twisted left- and right regular representation L^ω and R^ω are well-defined ω-representations of G.*

Proof. (We give the proof only for the left regular representation.) To show that L^ω is well-defined it suffices to show that for each $z \in G$ the operator L^ω_z is unitary (linearity is obvious). In particular, then, it is isometric, and so $\|L^\omega_z f\|_2 = \|f\|_2 < \infty$, giving immediately that $L^\omega_z f \in L^2(G)$. A calculation with the cocycle identity gives that the inverse of L^ω_z is given by $\overline{\omega(z^{-1}, z)} L^\omega_{z^{-1}}$ (this holds generally for ω-representations). Now for $f, g \in L^2(G)$:

$$\langle L^\omega_z f | g \rangle_2 = \int_G \overline{L^\omega_z f(x)} g(x) \,\mathrm{d}\mu(x) = \int_G \overline{\omega(z, z^{-1}x) f(z^{-1}x)} g(x) \,\mathrm{d}\mu(x) = \int_G \overline{\omega(z, x) f(x)} g(zx) \,\mathrm{d}\mu(x),$$

where we use left-invariance of the Haar measure in the last step. The cocycle identity gives $\omega(z^{-1}, z) = \omega(z^{-1}, zx)\omega(z, x)$ for all $x \in G$, so we substitute it into the above equation to find

$$\langle L^\omega_z f | g \rangle_2 = \int_G \overline{f(x)\omega(z^{-1}, z)} \omega(z^{-1}, zx) g(zx) \,\mathrm{d}\mu(x) = \int_G \overline{f(x)\omega(z^{-1}, z)} L^\omega_{z^{-1}} g(x) \,\mathrm{d}\mu(x) = \left\langle f \middle| \overline{\omega(z^{-1}, z)} L^\omega_{z^{-1}} \right\rangle_2,$$

which implies that $(L^\omega_z)^* = (L^\omega_z)^{-1}$, hence $L^\omega_z \in \mathrm{U}(L^2(G))$.

Now for the algebraic properties of L^ω. We need to verify that L^ω satisfies the twisted multiplication law: let $x, y, z \in G$ be group elements, and fix a function $f \in L^2(G)$. Unpacking the definitions:

$$L^\omega_z L^\omega_y f(x) = \omega(z, z^{-1}x)\omega(y, y^{-1}z^{-1}x) f(y^{-1}z^{-1}x).$$

On the other hand:

$$L^\omega_{zy} f(x) = \omega(zy, y^{-1}z^{-1}x) f(y^{-1}z^{-1}x),$$

which, combining with the cocycle relation $\omega(z, z^{-1}x)\omega(y, y^{-1}z^{-1}x) = \omega(z, y)\omega(zy, y^{-1}z^{-1}x)$ gives

$$L^\omega_z L^\omega_y f(x) = \omega(z, y) L^\omega_{zy} f(x),$$

meaning $L^\omega_z L^\omega_y = \omega(z, y) L^\omega_{zy}$. □

We verify some useful relations between the left- and right regular representations and the convolution:

Proposition 3.8. *Let $y \in G$ and $f, g \in C_\omega(G)$. Then, denoting by $\overline{\omega}$ the pointwise complex conjugate multiplier of ω, we have the following two identities:*

$$f *_\omega (L^\omega_y g) = \Delta(y^{-1}) \left((R^{\overline{\omega}}_y)^{-1} f \right) *_\omega g = \Delta(y^{-1}) \omega(y^{-1}, y) (R^{\overline{\omega}}_{y^{-1}} f) *_\omega g,$$

and

$$(L^\omega_y f)^\omega = \Delta(y) R^{\overline{\omega}}_y (f^\omega).$$

Proof. Using the cocycle identity and the definitions one easily verifies that

$$(f *_\omega (L^\omega_y g))(x) = \int_G \omega(z, y)\omega(zy, y^{-1}z^{-1}x) f(z) g(y^{-1}z^{-1}x) \,\mathrm{d}\mu(z).$$

Similarly, using the characteristic properties of the left Haar measure and modular function Δ, we find

$$\begin{aligned}\Delta(y^{-1})((R_y^{\overline{\omega}})^{-1}f *_\omega g)(x) &= \Delta(y^{-1})\int_G \omega(z,z^{-1}x)\omega(y,y^{-1})\overline{\omega(z,y^{-1})}f(zy^{-1})g(z^{-1}x)\,\mathrm{d}\mu(z)\\ &= \int_G \omega(zy,y^{-1}z^{-1}x)\omega(y,y^{-1})\overline{\omega(zy,y^{-1})}f(z)g(y^{-1}z^{-1}x)\,\mathrm{d}\mu(z).\end{aligned}$$

Invoking (2) for the multipliers in the integrand now gives the first equality. The second equality may be derived similarly and more simply. □

We calculate the integrated form of L^ω. By its construction via Lemma 3.5, we have that for all $f\in C_\omega(G)$ and $g,h\in L^2(G)$ that

$$\langle h|L_C^\omega(f)g\rangle_2 = \int_G \big\langle h\big|f(y)L_y^\omega g\big\rangle_2\,\mathrm{d}\mu(y) = \int_G\int_G \overline{h(z)}f(y)\omega(y,y^{-1}z)g(y^{-1}z)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y).$$

Employing Fubini's Theorem to switch the integration order, we get

$$\langle h|L_C^\omega(f)g\rangle_2 = \int_G \overline{h(z)}\int_G f(y)\omega(y,y^{-1}zg(y^{-1}z)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) = \int_G \overline{h(z)}(f*_\omega g)(z)\,\mathrm{d}\mu(z) = \langle h|f*_\omega g\rangle_2\,.$$

Therefore, the integrated form of the twisted left regular representation is simply twisted convolution:

Proposition 3.9. *For all $f,g\in C_\omega(G)$ we have $L_C^\omega(f)g = f*_\omega g$.*

Stepping back to the original twisted left regular representation L^ω, we have the following interaction with integrated forms of other ω-representations:

Lemma 3.10. *Let $\pi: G\to \mathrm{U}(\mathscr{H})$ be an ω-representation, and let $L^\omega : G\to \mathrm{U}(L^2(G))$ be the corresponding twisted left regular representation. Then the integrated form of π satisfies*

$$\pi_C(L_z^\omega f) = \pi(z)\pi_C(f).$$

Proof. This relation follows directly from the definitions and left invariance of the Haar measure: let $\xi,\eta\in\mathscr{H}$, $z\in G$ and $f\in C_\omega(G)$. Then

$$\begin{aligned}\langle\eta|\pi_C(L_z^\omega f)\xi\rangle &= \int_G \langle\eta|L_z^\omega f(x)\pi(x)\xi\rangle\,\mathrm{d}\mu(x) = \int_G \big\langle\eta\big|\omega(z,z^{-1}x)f(z^{-1}x)\pi(x)\xi\big\rangle\,\mathrm{d}\mu(x)\\ &= \int_G \langle\eta|f(x)\omega(z,x)\pi(zx)\xi\rangle\,\mathrm{d}\mu(x) = \int_G \langle\eta|f(x)\pi(z)\pi(x)\xi\rangle\,\mathrm{d}\mu(x)\\ &= \langle\eta|\pi(z)\pi_C(f)\xi\rangle\,.\end{aligned}$$

□

Using this result, we may finally prove our claim:

Proposition 3.11. *Given an ω-representation $\pi: G\to\mathrm{U}(\mathscr{H})$, its integrated form $\pi_C : C_\omega(G)\to \mathrm{B}(\mathscr{H})$ is a $*$-homomorphism.*

Proof. Linearity is clear from the construction of the integrated forms, so we are left to show that they are multiplicative and respect the involution.

Take $\xi,\eta\in\mathscr{H}$, and let $f,g\in C_\omega(G)$. First, by definition of the integrated form, we have

$$\langle\eta|\pi_C(f*_\omega g)\xi\rangle = \int_G\langle\eta|(f*_\omega g)(x)\pi(x)\xi\rangle\,\mathrm{d}\mu(x) = \int_G\int_G \big\langle\eta\big|\omega(y,y^{-1}x)f(y)g(y^{-1}x)\pi(x)\xi\big\rangle\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x).$$

In the integrand we recognise $\omega(y, y^{-1}x)g(y^{-1}x) = L_y^\omega g(x)$, so that by using Fubini's Theorem we obtain an integral

$$\int_G \langle \eta | f(y) L_y^\omega g(x) \pi(x) \xi \rangle \, \mathrm{d}\mu(x) = \langle \eta | f(y) \pi_C(L_y^\omega g) \xi \rangle \,,$$

to which we can apply the result of Lemma 3.10. Having done this, and substituting back into the original integral, we find

$$\langle \eta | \pi_C(f *_\omega g) \xi \rangle = \int_G \langle \eta | f(y) \pi(y) \pi_C(g) \xi \rangle \, \mathrm{d}\mu(y) = \langle \eta | \pi_C(f) \pi_C(g) \xi \rangle \,.$$

Thus: $\pi_C(f *_\omega g) = \pi_C(f)\pi_C(g)$.

For the involution we further calculate

$$\langle \eta | \pi_C(f^\omega) \xi \rangle = \int_G \left\langle \eta \middle| \overline{\omega(x, x^{-1})} \Delta(x^{-1}) \overline{f(x^{-1})} \pi(x) \xi \right\rangle \mathrm{d}\mu(x) = \int_G \left\langle \eta \middle| \overline{\omega(x^{-1}, x) f(x)} \pi(x^{-1}) \xi \right\rangle \mathrm{d}\mu(x),$$

using the characteristic properties of the modular function. Writing $\pi(x^{-1}) = \omega(x^{-1}, x)\pi(x)^{-1}$ we find

$$\langle \eta | \pi_C(f^\omega) \xi \rangle = \int_G \left\langle \eta \middle| |\omega(x^{-1}, x)|^2 \, \overline{f(x)} \pi(x)^* \xi \right\rangle \mathrm{d}\mu(x) = \int_G \left\langle \eta \middle| \overline{f(x)} \pi(x)^* \xi \right\rangle \mathrm{d}\mu(x) = \langle \eta | \pi_C(f)^* \xi \rangle \,,$$

so that indeed $\pi_C(f^\omega) = \pi_C(f)^*$, as desired. □

Recall that a $*$-representation $\rho : A \to \mathrm{B}(\mathscr{H})$ of some C*-algebra A is called ***nondegenerate*** whenever the space $\rho(A)\mathscr{H} = \operatorname{span}\{\rho(a)\xi : a \in A, \xi \in \mathscr{H}\}$ is dense in $\mathscr{H}$. If $(e_\iota)_{\iota \in I}$ is an approximate unit in A, nondegeneracy is equivalent to the equation $\xi = \lim_{\iota \in I} \rho(e_\iota)\xi$ holding for all $\xi \in \mathscr{H}$. (See [AC18, Definition 3, 25-09-2017].)

All integrated forms are in fact nondegenerate:

Lemma 3.12. *Let $\pi : G \to \mathrm{U}(\mathscr{H})$ be an ω-representation of G. Then its integrated form $\pi_C : C_\omega(G) \to \mathrm{B}(\mathscr{H})$ is nondegenerate (Cf. [AC18, Lemma 3, 16-10-2017].)*

Proof. It suffices to show that for every Dirac net $(f_i)_{i \in I}$ in $C_\omega(G)$ we have that $(\pi_C(f_i))_{i \in I}$ strongly converges to the identity operator on $\mathscr{H}$, i.e., so that for all $\xi \in \mathscr{H}$ we have the following convergence: $\xi = \lim_{i \in I} \pi_C(f_i)\xi$.

For arbitrary $\xi, \eta \in \mathscr{H}$, with $\eta \neq 0$, we have by the Cauchy-Schwarz inequality

$$|\langle \eta | \pi_C(f_i)\xi - \xi \rangle| \leqslant \int_G |\langle \eta | f_i(x)\pi(x)\xi - \xi \rangle| \, \mathrm{d}\mu(x) \leqslant \int_G \|\eta\| \, \|f_i(x)\pi(x)\xi - \xi\| \, \mathrm{d}\mu(x).$$

Noting that $\int_G \|f_i(x)\xi - \xi\| \, \mathrm{d}\mu(x) = 0$ by the fact that f_i is normalised on G, the triangle inequality moreover gives

$$|\langle \eta | \pi_C(f_i)\xi - \xi \rangle| \leqslant \int_G \|\eta\| \, f_i(x) \, \|\pi(x)\xi - \xi\| \, \mathrm{d}\mu(x) + \int_G \|\eta\| \, \|f_i(x)\xi - \xi\| \, \mathrm{d}\mu(x) = \int_G \|\eta\| \, f_i(x) \, \|\pi(x)\xi - \xi\| \, \mathrm{d}\mu(x).$$

For every $\varepsilon > 0$ there now exists an open neighbourhood $U \subseteq G$ of the identity element, such that for all $x \in U$ we have $\|\pi(x)\xi - \xi\| < \varepsilon / \|\eta\|$. In turn, there exists an index $i_0 \in I$ such that for all $i \geqslant i_0$ the support of f_i is contained in U. Therefore, for all such $i \geqslant i_0$ we obtain

$$|\langle \eta | \pi_C(f_i)\xi - \xi \rangle| = \int_U \|\eta\| \, f_i(x) \, \|\pi(x)\xi - \xi\| \, \mathrm{d}\mu(x) < \int_U \|\eta\| \, f_i(x)\varepsilon / \|\eta\| \, \mathrm{d}\mu(x) = \int_G \varepsilon f_i(x) \, \mathrm{d}\mu(x) = \varepsilon.$$

The result follows. □

Definition 3.13. For every continuous unitary representation $\pi : G \to \mathrm{U}(\mathscr{H})$ we define the seminorm $\|f\|_\pi := \|\pi_C(f)\|_{\mathrm{B}(\mathscr{H})}$ on the space $C_\omega(G)$. The ***(twisted) maximal C*-norm*** on $C_\omega(G)$ is defined as

$$\|f\| := \sup_{\pi \in \hat{G}_\omega} \|f\|_\pi = \sup_{\pi \in \hat{G}_\omega} \|\pi_C(f)\|_{\mathrm{B}(\mathscr{H})} .$$

The completion of $C_\omega(G)$ with respect to the maximal C*-norm is called the ***twisted group C*-algebra***, denoted $C^*_\omega(G)$. Note that the norm inherits the C*-identity from that of $\mathrm{B}(\mathscr{H})$, and from the fact that π_C is a $*$-representation of $C_\omega(G)$.

Lemma 3.14. *Let ω be a continuous multiplier on G. For fixed $x \in G$, the left twisted translation map $L^\omega_x : C_\omega(G) \to C_\omega(G)$ extends to an isometric linear map on $C^*_\omega(G)$. For fixed $f \in C^*_\omega(G)$ the map $L^\omega f : G \to C^*_\omega(G); y \mapsto L^\omega_y f$ is continuous.*

Proof. We note that for $f \in C_\omega(G)$ and $x \in G$, the maximal norm of the function $L^\omega_x f$ can be computed as follows:

$$\|L^\omega_x f\| = \sup_{\pi \in \hat{G}_\omega} \|\pi_C(L^\omega_x f)\| = \sup_{\pi \in \hat{G}_\omega} \|\pi(x)\pi_C(f)\| ,$$

using Lemma 3.10. But each $\pi(x)$ is a unitary representation on some Hilbert space, so $\|\pi(x)\pi_C(f)\| = \|\pi_C(f)\|$, which gives

$$\|L^\omega_x f\| = \sup_{\pi \in \hat{G}_\omega} \|\pi_C(f)\| = \|f\| .$$

This shows that L^ω_x is an isometric map on $C_\omega(G)$, and hence can be extended continuously to an isometric map to the entire space $C^*_\omega(G)$.

Following the proof of Lemma 9 in [AC18, 16-10-2017], we define

$$C := \{f \in C^*_\omega(G) : L^\omega f \text{ is continuous}\}.$$

We claim that this set is closed in $C^*_\omega(G)$. Let $(f_n)_{n\in\mathbb{N}}$ be some convergent sequence in C with limit $f \in C^*_\omega(G)$. We need to show that $f \in C$. For this, take a sequence $(x_n)_{n\in\mathbb{N}}$ in G, converging to $x \in G$. Repeatedly using the triangle inequality and isometry of L^ω_x gives an estimate

$$\left\|L^\omega_{x_m} f - L^\omega_x f\right\| \leqslant 2\,\|f_n - f\| + \left\|L^\omega_{x_m} f_n - L^\omega_x f_n\right\| ,$$

for all $m, n \in \mathbb{N}$, from which it is clear that $(L^\omega_{x_m})_{m\in\mathbb{N}}$ will converge to $L^\omega_x f$, hence showing that $L^\omega f$ is continuous, and proving that C is closed.

By construction of the twisted group C*-algebra it now suffices to show that C is contains the dense subspace $C_\omega(G)$. Fixing $f \in C_\omega(G)$, we claim that $L^\omega f$ is continuous with respect to the L^1-norm. Take again the converging sequence $(x_n)_{n\in\mathbb{N}}$ in G from the previous paragraph. Writing out the definitions, we have

$$\left\|L^\omega_{x_n} f - L^\omega_x f\right\|_1 = \int_G \left|L^\omega_{x_n} f(y) - L^\omega_x f(y)\right| \mathrm{d}\mu(y) = \int_G \left|\frac{\omega(x_n, x_n^{-1}y)}{\omega(x, x^{-1}y)} f(x_n^{-1}y) - f(x^{-1}y)\right| \mathrm{d}\mu(y).$$

The integrand can be estimated as follows:

$$\left|\frac{\omega(x_n, x_n^{-1}y)}{\omega(x, x^{-1}y)} f(x_n^{-1}y) - f(x^{-1}y)\right| \leqslant \left|f(x_n^{-1}y) - f(x^{-1}y)\right| + \left|f(x^{-1}y)\right| \left|\frac{\omega(x_n, x_n^{-1}y)}{\omega(x, x^{-1}y)} - 1\right| ,$$

which, by continuity of both the multiplier ω and of f, is seen to vanish when taking the limit over n. It follows that $(L^\omega_{x_n} f)_{n\in\mathbb{N}}$ converges to $L^\omega_x f$, showing that $L^\omega f$ is continuous in the L^1-norm. The fact that $f \in C$ now follows from the fact that the maximal C*-norm is bounded by the L^1-norm. ☐

Lemma 3.15. *Let $\rho : C^*_\omega(G) \to \mathrm{B}(\mathscr{H})$ be a nondegenerate $*$-representation. Then for all $f, g \in C_\omega(G)$ and $\xi, \eta \in \mathscr{H}$ we have*

$$\langle \eta | \rho(f *_\omega g)\xi\rangle = \int_G \langle \eta | f(x)\rho(L^\omega_x g)\xi\rangle \,\mathrm{d}\mu(x).$$

Proof. We follow the proof of [AC18, Lemma 1, 19-10-2017].

Throughout the proof, we fix $\varepsilon > 0$. Since for $g \in C_\omega(G)$ the map $L^\omega g$ is continuous with respect to the L^1-norm, for every x in the support of f we can find an open neighbourhood U_x of x such that for every other $y \in U_x$ we have $\left\|L^\omega_x g - L^\omega_y g\right\|_1 < \varepsilon$. The family $(U_x)_{x\in\mathrm{supp}(f)}$ forms an open cover for the support of f. By compactness we can find finitely many points $x_1, \dots, x_n \in \mathrm{supp}(f)$ such that $(U_{x_i})_{i=1}^n$ still covers $\mathrm{supp}(f)$. We now have a finite cover $(U_{x_i})_{i=1}^n \cup \{G \setminus \mathrm{supp}(f)\}$ of the entire group G. Since G is a locally compact second countable Hausdorff space, we can find a continuous partition of unity $(h_i)_{i=0}^n$ subordinate to this cover.

We write $f_i := h_i f$. Noting that $(f *_\omega g)(y) = \int_G f(x)L^\omega_x(y)\,\mathrm{d}\mu(x)$, we have a real number α that satisfies

$$\begin{aligned}\alpha &:= \left\| f *_\omega g - \sum_{i=1}^n \left(\int_G f_i(x)\,\mathrm{d}\mu(x)\right) L^\omega_{x_i} g \right\|_1 \\ &= \int_G \left| \int_G f(x)L^\omega_x g(y)\,\mathrm{d}\mu(x) - \sum_{i=1}^n \left(\int_G f_i(x)\,\mathrm{d}\mu(x)\right) L^\omega_{x_i} g(y)\right| \mathrm{d}\mu(y).\end{aligned}$$

By elementary manipulations and the fact that $f = \sum_{i=1}^n f_i$, we find

$$\alpha = \int_G \left| \int_G \sum_{i=1}^n f_i(x)\left(L^\omega_x g(y) - L^\omega_{x_i} g(y)\right)\mathrm{d}\mu(x)\right| \mathrm{d}\mu(y) \leqslant \int_G \sum_{i=1}^n |f_i(x)| \left\|L^\omega_x g - L^\omega_{x_i} g\right\|_1 \mathrm{d}\mu(x).$$

Now (switching the sum and integral) since f_i is supported on U_{x_i} we have per construction of the open cover that $\int_G |f_i(x)| \left\|L^\omega_x g - L^\omega_{x_i} g\right\| < \varepsilon \int_G |f_i(x)|$, so that we have the following inequality for α:

$$\alpha < \varepsilon \int_G \sum_{i=1}^n |f_i(x)|\,\mathrm{d}\mu(x) = \varepsilon \int_G \sum_{i=1}^n |h_i(x)|\,|f(x)|\,\mathrm{d}\mu(x) = \varepsilon \int_G |f(x)|\,\mathrm{d}\mu(x) = \varepsilon \|f\|_1 . \tag{3}$$

Using the triangle inequality, we approximate

$$\begin{aligned}&\left| \langle \eta | \rho(f *_\omega g)\xi\rangle - \int_G \langle \eta | f(x)\rho(L^\omega_x g)\xi\rangle \,\mathrm{d}\mu(x)\right| \\ &\leqslant \left| \langle \eta | \rho(f *_\omega g)\xi\rangle - \int_G \sum_{i=1}^n \left\langle \eta \middle| f_i(x)\rho(L^\omega_{x_i} g)\xi\right\rangle \mathrm{d}\mu(x)\right| \\ &+ \left| \int_G \sum_{i=1}^n \left\langle \eta \middle| f_i(x)\rho(L^\omega_{x_i} g)\xi\right\rangle \mathrm{d}\mu(x) - \int_G \langle \eta | f(x)\rho(L^\omega_x g)\xi\rangle \,\mathrm{d}\mu(x)\right|.\end{aligned} \tag{4}$$

Using (3) we will estimate each of the two terms on the right hand side individually.

For the first term, we first note that, since ρ is a $*$-homomorphism between C*-algebras, it is norm decreasing. In turn we have for all $h \in C_\omega(G)$ that $\|\rho(h)\|_{\mathrm{B}(\mathscr{H})} \leqslant \|h\| \leqslant \|h\|_1$. Using linearity of ρ we write the first term in the above equation as

$$\langle \eta | \rho(f *_\omega g)\xi\rangle - \int_G \sum_{i=1}^n \left\langle \eta \middle| f_i(x)\rho(L^\omega_{x_i} g)\xi\right\rangle \mathrm{d}\mu(x) = \left\langle \eta \middle| \rho\left(f *_\omega g - \sum_{i=1}^n \left(\int_G f_i(x)\,\mathrm{d}\mu(x)\right) L^\omega_{x_i} g\right) \xi \right\rangle,$$

so now the Cauchy-Schwartz inequality together with $\|\rho(h)\| \leqslant \|h\|_1$ and (3) give the following upper bound for the first term:

$$\|\eta\| \, \|\xi\| \left\| f *_\omega g - \sum_{i=1}^{n} \left(\int_G f_i(x) \, \mathrm{d}\mu(x) \right) L^\omega_{x_i} g \right\|_1 = \|\eta\| \, \|\xi\| \, \alpha < \varepsilon \, \|\eta\| \, \|\xi\| \, \|f\|_1 .$$

We calculate an estimate for the second term similarly (using the same inequalities): it has an upper bound

$$\|\eta\| \, \|\xi\| \int_G \left\| \sum_{i=1}^{n} f_i(x)\rho(L^\omega_{x_i} g) - f(x)\rho(L^\omega_x g) \right\| \mathrm{d}\mu(x) \leqslant \|\eta\| \, \|\xi\| \int_G \sum_{i=1}^{n} |f_i(x)| \, \|\rho(L^\omega_{x_i} g - L^\omega_x g)\| \, \mathrm{d}\mu(x)$$
$$< \varepsilon \, \|\eta\| \, \|\xi\| \int_G |f(x)| \, \mathrm{d}\mu(x) = \varepsilon \, \|\eta\| \, \|\xi\| \, \|f\|_1 .$$

Substituting back into (4), we find that for all $\varepsilon > 0$:

$$\left| \langle \eta | \rho(f *_\omega g)\xi \rangle - \int_G \langle \eta | f(x)\rho(L^\omega_x g)\xi \rangle \, \mathrm{d}\mu(x) \right| < 2\varepsilon \, \|\eta\| \, \|\xi\| \, \|f\|_1 ,$$

and the equality follows. □

We now describe the inverse procedure of integrating an ω-representation to the twisted group C*-algebra:

Lemma 3.16. *Let $\rho : C^*_\omega(G) \to \mathrm{B}(\mathscr{H})$ be a nondegenerate $*$-representation. For $x \in G$ fixed, we define on $\rho(C^*_\omega(G))\mathscr{H} = \operatorname{span}\{\rho(f)\xi : f \in C^*_\omega(G), \xi \in \mathscr{H}\}$ the following map:*

$$\widetilde{\rho}_G(x) : \rho(C^*_\omega(G))\mathscr{H} \to \mathscr{H}$$
$$\sum_{i=1}^{n} \rho(f_i)\xi_i \mapsto \sum_{i=1}^{n} \rho(L^\omega_x f_i)\xi_i .$$

This defines an ω-representation $\rho_G : G \to \mathrm{U}(\mathscr{H})$, such that its integrated form satisfies $(\rho_G)_C = \rho$.

Proof. First we must show that $\widetilde{\rho}_G(x)$ is well-defined on the span $\rho(C^*_\omega(G))\mathscr{H}$. Let $f_1, \ldots, f_n,\, g_1, \ldots, g_m \in C^*_\omega(G)$ and $\xi_1, \ldots, \xi_n,\, \eta_1, \ldots, \eta_m \in \mathscr{H}$ such that $\sum_{l=1}^{n} \rho(f_l)\xi_l = \sum_{k=1}^{m} \rho(g_k)\eta_k$. Let $(h_\iota)_{\iota \in I}$ be an approximate unit in $C^*_\omega(G)$. Using the first identity in Proposition 3.8 we calculate:

$$L^\omega_x f_l = \lim_{\iota \in I} h_\iota (L^\omega_x f_l) = \lim_{\iota \in I} \Delta(x^{-1})((R^{\overline{\omega}}_x)^{-1} h_\iota) f_l ,$$

for each $l \in \{1, \ldots, n\}$. By continuity of ρ we have therefore $\rho(L^\omega_x f_l) = \lim_{\iota \in I} \Delta(x^{-1})\rho((R^{\overline{\omega}}_x)^{-1} h_\iota)\rho(f_l)$, so that $\widetilde{\rho}_G(x)$ maps $\sum_{l=1}^{n} \rho(f_l)\xi_l$ to:

$$\sum_{l=1}^{n} \rho(L^\omega_x f_l)\xi_l = \sum_{l=1}^{n} \lim_{\iota \in I} \Delta(x^{-1})\rho((R^{\overline{\omega}}_x)^{-1} h_\iota)\rho(f_l)\xi_l = \lim_{\iota \in I} \Delta(x^{-1})\rho((R^{\overline{\omega}}_x)^{-1} h_\iota) \sum_{l=1}^{n} \rho(f_l)\xi_l$$
$$= \lim_{\iota \in I} \Delta(x^{-1})\rho((R^{\overline{\omega}}_x)^{-1} h_\iota) \sum_{k=1}^{m} \rho(g_k)\eta_k = \sum_{k=1}^{m} \rho(L^\omega_x g_k)\eta_k ,$$

showing that it is well-defined.

Now using that ρ is a $*$-homormorphism, and both identities in Proposition 3.8 (firstly the second, and then the first), we find that for every $\xi, \eta \in \mathscr{H}$ and $f, g \in C^*_\omega(G)$:

$$\langle \rho(L^\omega_x f)\xi | \rho(L^\omega_x g)\eta \rangle = \langle \xi | \Delta(x)\rho\left(R^{\overline{\omega}}_x(f^\omega)\right)\rho(L^\omega_x g)\eta \rangle = \left\langle \xi \middle| \Delta(x)\rho\left(\Delta(x^{-1})\left(R^{\overline{\omega}}_x\right)^{-1} R^{\overline{\omega}}_x(f^\omega) g\right)\eta \right\rangle$$
$$= \langle \xi | \rho(f^\omega)\rho(g)\eta \rangle = \langle \rho(f)\xi | \rho(g)\eta \rangle .$$

By linearity of the inner product and nondegeneracy of ρ it follows that $\widetilde{\rho}_G(x)$ extends to a unitary map $\rho_G(x)$ on the entire Hilbert space $\mathscr{H}$. It is straightforward to verify that the induced map $\rho_G : G \to \mathrm{U}(\mathscr{H})$ has the multiplicativity property of an ω-representation. To show that ρ_G is continuous we need to show that for every sequence $(x_n)_{n\in\mathbb{N}}$ in G converging to some $x \in G$, we have the convergence $\rho(L^\omega_{x_n} f)\xi \to \rho(L^\omega_x f)\xi$ in the Hilbert space. This is evident from the continuity of ρ and the continuity of $L^\omega f$ (see Lemma 3.14).

Hence we have constructed the continuous ω-representation $\rho_G : G \to \mathrm{U}(\mathscr{H})$, and we are left to show that its integrated form returns ρ. For this we use Lemma 3.15: on elements $\rho(g)\xi$ of the dense subspace $\rho(C^*_\omega(G))\mathscr{H}$ we have

$$\langle\eta|(\rho_G)_C(f)\rho(g)\xi\rangle = \int_G \langle\eta|f(x)\rho_G(x)\rho(g)\xi\rangle \,\mathrm{d}\mu(x) = \int_G \langle\eta|f(x)\rho(L^\omega_x g)\xi\rangle \,\mathrm{d}\mu(x) = \langle\eta|\rho(f *_\omega g)\xi\rangle = \langle\eta|\rho(f)\rho(g)\xi\rangle\,.$$

It follows that $(\rho_G)_C(f)$ equals $\rho(f)$ in the entire Hilbert space $\mathscr{H}$, and hence $(\rho_G)_C = \rho$. □

Lemma 3.17. *Let $\pi : G \to \mathrm{U}(\mathscr{H})$ be an ω-representation. Then $(\pi_C)_G = \pi$.*

Proof. Since π_C is nondegenerate (by Lemma 3.12) it suffices to show that $(\pi_C)_G(x) = \pi(x)$ on the dense subspace $\pi_C(C^*_\omega(G))\mathscr{H}$, for all $x \in G$. This follows by Lemma 3.10:

$$\langle\eta|\pi(x)\pi_C(f)\xi\rangle = \langle\eta|\pi_C(L^\omega_x f)\xi\rangle = \langle\eta|(\pi_C)_G(x)\pi_C(f)\xi\rangle\,.$$ □

Combining the previous two lemmas, we have the main result of this section:

Theorem 3.18. *Let ω be a continuous multiplier on G. There is a bijective correspondence between continuous ω-representations of G, and nondegenerate $*$-representations of $C^*_\omega(G)$.*

4 The twisted version of the Peter-Weyl theorem

For compact topological groups there is the well-known result that irreducible unitary representations are necessarily finite dimensional, and that any unitary representation can be decomposed as a direct sum of irreducible ones (see [Fol94, Theorem 5.2]). These facts are summarised in the following theorem:

Theorem 4.1 (Peter-Weyl)**.** *Let G be a compact topological group. The (ordinary) left regular representation $L : G \to \mathrm{U}(L^2(G))$ on the Hilbert space of square integrable functions is isomorphic to the direct sum of all irreducible unitary representations:*

$$L \cong \bigoplus_{[\pi]\in\hat{G}} \pi^{d_\pi}.$$

Here $d_\pi := \dim(\mathscr{H}_\pi) \in \mathbb{N}$ and π^{d_π} denotes the d_π-fold direct sum of π. (See [Fol94, Theorem 5.12].)

The Peter-Weyl theorem can be restated (via Theorem 3.18) in the langauge of group C*-algebras as follows:

Proposition 4.2. *Let G be a compact group, and let $C^*(G)$ denote its ordinary group C*-algebra (corresponding to the twisted group C*-algebra with $\omega = 1$). Then*

$$C^*(G) \cong \bigoplus_{[\pi]\in\hat{G}} \mathrm{M}_{d_\pi}(\mathbb{C}),$$

where again $d_\pi := \dim(\mathscr{H}_\pi) \in \mathbb{N}$ is the dimension of the Hilbert space corresponding to the irreducible representation $\pi : G \to \mathrm{U}(\mathscr{H}_\pi)$, and $\mathrm{M}_{d_\pi}(\mathbb{C})$ denotes the C-algebra of complex $d_\pi \times d_\pi$ matrices. (See [Wil07, Proposition 3.4].)*

It turns out that an analogue of this result is also true for twisted C*-algebras. We refer to [Lan98, Section III.1.8] for a treatment of the Peter-Weyl theorem in the context of twisted group C*-algebras of Lie groups with smooth multiplier.

4.1 Irreducible ω-representations on compact groups

One part of the Peter-Weyl theorem is the result that irreducible unitary representations of compact groups are necessarily finite dimensional. We sketch an argument for the analogous claim for projective representations. In this, we follow the proofs in the notes [AC18].

First, given an ω-representation $\pi : G \to \mathrm{U}(\mathscr{H})$, we define the so called *matrix coefficient* $\phi_{\eta,\xi} \in C(G)$ by

$$\phi_{\eta,\xi}(x) := \langle \pi(x)^{-1}\eta | \xi \rangle = \omega(x, x^{-1}) \langle \pi(x^{-1})\eta | \xi \rangle ,$$

for each $\xi, \eta \in \mathscr{H}$. A straightforward calculation shows that, due to the compactness of G, the matrix coefficients are $L^2(G)$ functions:

$$\|\phi_{\eta,\xi}\|_2^2 = \int_G \left\| \langle \pi(x)^{-1}\eta | \xi \rangle \right\|_2^2 \mathrm{d}\mu(x) \leqslant \|\xi\|^2 \|\eta\|^2 \int_G \mathrm{d}\mu(x) < \infty.$$

From now on, we shall normalise the Haar measure μ on G such that $\int_G \mathrm{d}\mu(x) = 1$.

Now: the following lemma shows that any irreducible ω-representation

Lemma 4.3. *Let $\pi : G \to \mathrm{U}(\mathscr{H})$ be an irreducible ω-representation of a compact group G. Then there is an intertwiner $t : \mathscr{H} \to L^2(G)$ between π and the twisted left regular representation $L^\omega : G \to \mathrm{U}(L^2(G))$. That is, for all $x \in G$ we have $t \circ \pi(x) = L^\omega_x \circ t$.*

Proof. First, fixing a non-zero vector $\xi \in \mathscr{H}$, we define an operator $t : \mathscr{H} \to L^2(G)$, by $t(\eta) := \phi_{\eta,\xi}$. Due to the normalisation of the Haar measure it follows that t is a bounded linear operator with $\|t\| \leqslant \|\xi\|$. A straightforward calculation with (2) shows that t intertwines π and L^ω:

$$(t \circ \pi(y)\eta)(x) = \phi_{\pi(y)\eta,\xi}(x) = \omega(x, x^{-1})\overline{\omega(x^{-1}, y)}\phi_{\eta,\xi}(y^{-1}x) = L^\omega_y \phi_{\eta,\xi}(x) = (L^\omega_y \circ t(\eta))(x).$$

This equation holds for any two $x, y \in G$, so $t \circ \pi(y) = L^\omega_y \circ t$ for all $y \in G$. □

In conjunction with [Lan98, Proposition III.1.5.1], Schur's Lemma shows that t can be rescaled to an isometric map. It will follow that (cf. [AC18, Corollary 4, 13-11-2017])

Proposition 4.4. *If $\pi : G \to \mathrm{U}(\mathscr{H})$ is an irreducible ω-representation and G is compact, then* $\dim(\mathscr{H}) < \infty$.

4.2 Direct sum decomposition of the twisted group C*-algebra

Foremost we note that all compact groups are unimodular, so that the left Haar measure μ on G will also be right invariant. This fact immediately comes into play to show that the integrated form of the twisted left regular representation actually takes values in the space of compact operators on $L^2(G)$:

Lemma 4.5. *For every $f \in C^*_\omega(G)$ the operator $L^\omega_C(f) \in \mathrm{B}(L^2(G))$ is compact. The same is true for the twisted right regular representation.*

Proof. This is the proof of [Wil07, Lemma 3.5]. We saw in Proposition 3.9 that for $f, g \in C_\omega(G)$ the integrated form acts as $L^\omega_C(f)g = f *_\omega g$. Using the fact that G is unimodular, evaluation at some point $x \in G$ returns

$$L^\omega_C(f)g(x) = \int_G \omega(z, z^{-1}x)f(z)g(z^{-1}x)\,\mathrm{d}\mu(z) = \int_G \omega(xz^{-1}, z)f(xz^{-1})g(z)\,\mathrm{d}\mu(z).$$

It is clear that the function $(x, z) \mapsto \omega(xz^{-1}, z)f(xz^{-1})$ in the integrand is in $L^2(G \times G)$. Therefore, as an integral operator, $L^\omega_C(f)$ is Hilbert-Schmidt by [Mac08, Theorem 4.16]. But Hilbert-Schmidt operators on separable Hilbert spaces are compact (Theorem 4.15 of the same reference), so the result follows by denseness of $C_\omega(G)$ in $C^*_\omega(G)$. □

This lemma, together with the appropriate generalisations of [Wil07, Lemmas 3.6-7], will give that

Theorem 4.6. *Let G be a compact topological group, and let ω be a continuous multiplier on G. Then the twisted group C^*-algebra $C^*_\omega(G)$ decomposes into a direct sum:*

$$C^*_\omega(G) \cong \bigoplus_{[\pi]\in\hat{G}_\omega} \mathrm{M}_{d_\pi}(\mathbb{C}),$$

where the isomorphism is given by the map $f \mapsto (\pi_C(f))_{[\pi]\in\hat{G}_\omega}$. (Cf. [Lan98, Theorem III.1.8.1].)

References

[AC18] F. Arici and T. Crisp, *C*-algebras and representation theory*, 2018, Lecture notes, http://www.math.ru.nl/~tcrisp/teaching/2017-Cstar-reps/.

[EL69] C.M. Edwards and J.T. Lewis, *Twisted group algebras I*, Communications in Mathematical Physics **13** (1969), no. 2, 119–130.

[Fol94] G.B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, Taylor & Francis, 1994.

[Lan98] N.P. Landsman, *Mathematical topics between classical and quantum mechanics*, Springer Monographs in Mathematics, Springer New York, 1998.

[Mac58] G.W. Mackey, *Unitary representations of group extensions. I*, Acta Mathematica **99** (1958), no. 1, 265–311.

[Mac08] B. MacCluer, *Elementary functional analysis*, Graduate Texts in Mathematics, Springer New York, 2008.

[Men17] E.M. Mendonca, *Projective representations of groups*, 2017.

[RA15] H. Reinhard and R. Alfred, *Photons in fock space and beyond (in 3 volumes)*, World Scientific Publishing Company, 2015.

[Ros94] J. Rosenberg, *C*-algebras and Mackey's theory of group representations*, Contemp. Math **169** (1994), 150.

[Sch08] M. Schottenloher, *A mathematical introduction to conformal field theory*, Springer Berlin Heidelberg, 2008.

[Wig31] E.P. Wigner, *Gruppentheorie und ihre anwendung auf die quantenmechanik der atomspektren.*, Die Wissenschaft, Vieweg, 1931.

[Wil07] D.P. Williams, *Crossed products of C*-algebras*, no. 134, American Mathematical Soc., 2007.